**First MATH 4512 mid-term exam**: Sept. 28, 2015 NAME:

(Each problem is 4 points)

1. Find the solution of the initial-value problem $\frac{d}{dt}y - 2\,t\,y = t$ and $y(0) = 1$.

We are going to solve this problem by using the technique of the integrating factor $\mu(t)$. Such factor must be such that $\frac{\mu'(t)}{\mu(t)} = -2t$. Hence, we take $\mu(t) = e^{-t^2}$. Multiplying our equation by $\mu(t)$, we get that

$$e^{-t^2}\frac{d}{dt}y - 2t\,e^{-t^2}y = t\,e^{-t^2}$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{-t^2}y\right) = t\,e^{-t^2}$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{-t^2}y\right) = -\frac{1}{2}\frac{d}{dt}e^{-t^2}$$

Integrating from "0" to "t", we get

$$e^{-t^2}y(t) - y(0) = -\frac{1}{2}\left(e^{-t^2} - 1\right)$$

$$\Rightarrow \quad y(t) = y(0)\,e^{t^2} - \frac{1}{2}\left(1 - e^{t^2}\right)$$

and since $y(0) = 1$,

$$y(t) = \frac{3}{2}e^{t^2} - \frac{1}{2}.$$

2. Find the solution of the initial-value problem $\frac{d}{dt}y = \frac{2t}{y+yt^2}$ and $y(2) = 3$.

Multiplying the equation by y, we get

$$y \frac{d}{dt} y = \frac{2t}{1+t^2}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2} y^2\right) = \frac{d}{dt} \ln(1+t^2)$$

Integrating from "2" to "t", we get

$$\frac{1}{2} y^2(t) - \frac{1}{2} y^2(2) = \ln(1+t^2) - \ln(1+4)$$

$$\Rightarrow y^2(t) = y^2(2) + 2\ln\left(\frac{1+t^2}{5}\right)$$

$$\Rightarrow y(t) = \pm\left(9 + 2\ln\left(\frac{1+t^2}{5}\right)\right)^{1/2}$$

since $y(2) = 3$. Of the two possible solutions, we must take the positive one given that $y(2) = 3 > 0$. So,

$$y(t) = \left(9 + 2\ln\left(\frac{1+t^2}{5}\right)\right)^{1/2}.$$

3. State the theorem of the existence and uniqueness of the initial-value problem $\frac{d}{dt}y = f(y,t)$ and $y(t_0) = y_0$. When $f(y,t) = |y|^{1/2}, t_0 = 0$ and $y_0 = 0$ show that there are two solutions and argue why this does not contradict the theorem.

Assume that $f$ and $\frac{\partial}{\partial y} f$ are continuous functions on the rectangle $[y_0 - b, y_0 + b] \times [t_0, t_0 + a]$. Then there is a unique solution $y$ of the initia-value problem

$$(*) \qquad \frac{d}{dt} y = f(y,t), \quad y(t_0) = y_0$$

for $t \in [t_0, t_0 + \alpha]$, where $\alpha = \min\{a, b/M\}$,

where $M = \max\limits_{\substack{t_0 \le t \le t_0 + a \\ |y - y_0| \le b}} |f(y,t)|$.

If $f(y,t) = |y|^{1/2}$, $y(t) \equiv 0$ is a solution of $(*)$. Moreover, $y(t) = \frac{t^2}{4}$ is also a solution of $(*)$ for $t > 0$. This does not contradict the above result because it is impossible to find $b$ such that $f$ and $\frac{\partial}{\partial y} f$ are continuous functions on a rectangle of the form $[y_0 - b, y_0 + b] \times [t_0, t_0 + a]$ for $y_0 = 0$ and $t_0 = 0$. The Reason is that $\frac{\partial}{\partial y} f = \frac{1}{2} |y|^{-1/2}$ is unbounded at $y = y_0 = 0$.

4. Find the solution of the initial-value problem $9\frac{d^2}{d^2t}y - 12\frac{d}{dt}y + 4y = 0$, $y(\pi) = 0$ and $\frac{d}{dt}y(\pi) = 2$.

We know that the general solution is of the form

$$Y(t) = C_1 Y_1(t) + C_2 Y_2(t)$$

where $Y_1$ and $Y_2$ are two linearly independent solutions.

Let us find them. We assume that they are of the form $e^{rt}$. Inserting this form in the equation we get that

$$(9r^2 - 12r + 4)\, e^{rt} = 0 \implies (3r-2)^2 e^{rt} = 0$$

$$\implies r = \frac{2}{3}.$$

Hence $Y_1 = e^{\frac{2}{3}t}$. The other solution is not of the form $e^{rt}$ because we got repeated values of $r$. Hence $Y_2 = t\, e^{\frac{2}{3}t}$. This means that

$$Y(t) = C_1 e^{\frac{2}{3}t} + C_2 t\, e^{\frac{2}{3}t}$$
$$= (C_1 + C_2 t)\, e^{\frac{2}{3}t}.$$

Let us find $C_1$ and $C_2$. We have that

$$0 = Y(\pi) = (C_1 + C_2\pi)\, e^{\frac{2}{3}\pi}$$
$$2 = \frac{d}{dt}Y(\pi) = \left(C_1 \frac{2}{3} + C_2 \frac{2}{3}\pi + C_2\right) e^{\frac{2}{3}\pi}$$

$$\implies \quad 0 = C_1 + \pi C_2$$
$$2e^{-\frac{2}{3}\pi} = \frac{2}{3}C_1 + \left(\frac{2}{3}\pi + 1\right) C_2$$

$$\implies \quad C_1 = -\pi C_2 \text{ and } C_2 = 2e^{-\frac{2}{3}\pi} \implies C_1 = -2\pi e^{-\frac{2}{3}\pi}.$$

Hence

$$Y(t) = -2\pi\, e^{\frac{2}{3}(t-\pi)} + 2t\, e^{\frac{2}{3}(t-\pi)}$$
$$= 2(t-\pi)\, e^{\frac{2}{3}(t-\pi)}.$$

5. Find the solution of the initial-value problem $\frac{d^2}{d^2t}y - 3\frac{d}{dt}y + 2y = \sqrt{1+t}$, $y(0) = 0$ and $\frac{d}{dt}y(0) = 0$.

The general form of the solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \psi(t),$$

where $y_1$ and $y_2$ are two linearly independent solutions of the homogeneous problem and $\psi$ is a particular solution.

Let us find the solutions of the homogeneous equation. They are of the form $e^{rt}$, and so we must have that

$$(r^2 - 3r + 2)e^{rt} = 0 \quad \Rightarrow \quad r = 1 \text{ or } r = 2.$$

Hence $y_1(t) = e^{t}$ and $y_2(t) = e^{2t}$.

Now let us find the particular solution $\psi$. It is of the form

$$\psi(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

where

$$\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{1+t} \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} e^{t} & e^{2t} \\ e^{t} & 2e^{2t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{1+t} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = e^{-3t} \begin{bmatrix} 2e^{2t} & -e^{2t} \\ -e^{t} & e^{t} \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{1+t} \end{bmatrix} = e^{-3t} \begin{bmatrix} -\sqrt{1+t}\, e^{2t} \\ +\sqrt{1+t}\, e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} -\sqrt{1+t}\, e^{-t} \\ +\sqrt{1+t}\, e^{-2t} \end{bmatrix}$$

$$\Rightarrow \quad u_1(t) = -\int_0^t \sqrt{1+s}\, e^{-s}\, ds$$

$$u_2(t) = +\int_0^t \sqrt{1+s}\, e^{-2s}\, ds.$$

Let us find the constants $c_1$ and $c_2$. We have

$$0 = y(0) = c_1 y_1(0) + c_2 y_2(0) + \psi(0)$$
$$0 = \frac{d}{dt}y(0) = c_1 y_1'(0) + c_2 y_2'(0) + \psi'(0)$$

Since

$$y_1(0) = 1, \quad y_1'(0) = 1, \quad u_1(0) = 0, \quad u_1'(0) = -1$$
$$y_2(0) = 1, \quad y_2'(0) = 2, \quad u_2(0) = 0, \quad u_2'(0) = +1$$

$$\psi(0) = u_1(0)\, y_1(0) + u_2(0)\, y_2(0)$$
$$\frac{d}{dt}\psi(0) = u_1'(0)\, y_1(0) + u_2'(0)\, y_2(0) + u_1(0)\, y_1'(0) + u_2(0)\, y_2'(0)$$

we get that

$$\psi(0) = 0, \quad \frac{d}{dt}\psi(0) = 0$$

and that

$$\begin{aligned} 0 &= c_1 + c_2 \\ 0 &= c_1 + 2c_2 \end{aligned} \quad \Rightarrow c_1 = c_2 = 0$$

Hence

$$y(t) = \int_0^t \left(-\sqrt{1+s}\, e^{t-s} + \sqrt{1+s}\, e^{2(t-s)}\right) ds .$$