Second MATH 4512 mid-term exam: Oct. 23, 2015 NAME:

(Each problem is 4 points)

1. Find the Laplace transform of $\cos at$ and $\sin at$.

We know that $\mathcal{L}\{\cos at\} = \mathrm{Re}\,\mathcal{L}\{e^{iat}\}$ and that $\mathcal{L}\{\sin at\} = \mathrm{Im}\,\mathcal{L}\{e^{iat}\}$. But, by definition,

$$\mathcal{L}\{e^{iat}\} = \lim_{A\to\infty} \int_0^A e^{-st} e^{iat}\,dt = \lim_{A\to\infty} \frac{e^{-sA}e^{iaA} - 1}{-s+ia}$$

$$= \begin{cases} \frac{1}{s-ia} & \text{if } s>0, \\ \text{undefined} & \text{otherwise}. \end{cases}$$

Then $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

(for $s>0$).

2. Find the function whose Laplace transform is $s/(s^2+4)^2$. (Hint: Use the identity $\mathcal{L}\{-t\,f(t)\} = \frac{d}{ds}F(s)$.)

We have that

$$\frac{s}{(s^2+4)^2} = -\frac{1}{2}\frac{d}{ds}\frac{1}{(s^2+4)}.$$

Since $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$, by the previous problem, we have that

$$\frac{s}{(s^2+4)^2} = -\frac{1}{4}\frac{d}{ds}\mathcal{L}\{\sin 2t\}$$

$$= -\frac{1}{4}\mathcal{L}\{-t\sin 2t\}$$

$$= \mathcal{L}\left\{\frac{t}{4}\sin 2t\right\}.$$

Hence, $\frac{t}{4}\sin 2t$ is the function we want.

4. Using the Laplace transform, find the solution of the initial-value problem $\frac{d^2}{dt^2}y + 4y = \begin{cases} 1 & 0 \le t \le 4 \\ 0 & 4 < t \end{cases}$ and $y(0) = 3$ and $y'(0) = -2$. (Hint: Use the identity $\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \mathcal{L}\{(f * g)(t)\}$.)

Applying the Laplace transform to our equation and calling $f(t)$ its right hand side, we get

$$Y(s) = \frac{1}{s^2+4}\left((3s-2) + \mathcal{L}\{f(t)\}\right)$$

$$= 3\left(\frac{s}{s^2+4}\right) - \frac{2}{s^2+4} + \frac{1}{s^2+4}\cdot \mathcal{L}\{f(t)\}$$

$$= 3\,\mathcal{L}\{\cos 2t\} - \mathcal{L}\{\sin 2t\} + \mathcal{L}\{y_p(t)\}$$

where

$$y_p(t) = \frac{1}{2}\int_0^t f(\tau)\sin 2(t-\tau)\,d\tau$$

$$= \frac{1}{2}\begin{cases} \int_0^t \sin 2(t-\tau)\,d\tau & \text{if } t \le 4 \\ \int_0^4 \sin 2(t-\tau)\,d\tau & \text{if } t \geq 4 \end{cases}$$

$$= \frac{1}{2}\begin{cases} \frac{1}{2}\left[1 - \cos 2t\right] & \text{if } t \le 4 \\ \frac{1}{2}\left[\cos 2(t-4) - \cos 2t\right] & \text{if } t \geq 4 \end{cases}$$

$$= \frac{1}{4}\cos 2\max(t-4,0) - \frac{1}{4}\cos 2t.$$

This implies that

$$Y(t) = 3\cos 2t - \sin 2t + \frac{1}{4}\cos(2\max(t-4,0)) - \frac{1}{4}\cos 2t.$$

$$= \frac{11}{4}\cos 2t - \sin 2t + \frac{1}{4}\cos(2\max(t-4,0)).$$

3. Using the Laplace transform, find the solution of the initial-value problem $\frac{d^2}{dt^2}y + 4y = \cos 2t$ and $y(0) = y'(0) = 0$.

Applying the Laplace transform, we get

$$Y(s) = \frac{1}{s^2+4} \cdot \mathcal{L}\{\cos 2t\}$$

$$= \frac{1}{s^2+4} \cdot \frac{s}{s^2+4} \quad \text{by problem 1}$$

$$= \frac{s}{(s^2+4)^2}$$

$$= \mathcal{L}\left\{\frac{t}{4} \sin 2t\right\} \quad \text{by problem 2}$$

and so $y(t) = \frac{t}{4} \sin 2t$.

5. Using the Laplace transform, find the solution of the initial-value problem $\frac{d^2}{dt^2}y - 2\frac{d}{dt}y + y = t\,e^t$ and $y(0) = y'(0) = 0$.

Applying the Laplace transform, we get

$$Y(s) = \frac{1}{s^2-2s+1} \cdot \mathcal{L}\{t e^t\}$$

$$= \frac{1}{(s-1)^2} \frac{d}{ds} \mathcal{L}\{-e^t\}$$

$$= \frac{1}{(s-1)^2} \frac{d}{ds}\left(\frac{-1}{s-1}\right)$$

$$= \frac{1}{(s-1)^4}$$

$$= -\frac{1}{6} \frac{d^3}{ds^3}\left(\frac{1}{s-1}\right)$$

$$= -\frac{1}{6} \frac{d^3}{ds^3} \mathcal{L}\{e^t\}$$

$$= -\frac{1}{6} \frac{d^2}{ds^2} \mathcal{L}\{-t e^t\}$$

$$= -\frac{1}{6} \frac{d}{ds} \mathcal{L}\{t^2 e^t\}$$

$$= -\frac{1}{6} \mathcal{L}\{-t^3 e^t\}$$

$$= \mathcal{L}\left\{\frac{t^3}{6} e^t\right\}$$

and so $y(t) = \frac{1}{6} t^3 e^t$.