Third MATH 4512 mid-term exam: Nov. 23, 2015 NAME:

(Each problem is 4 points)

1. Find the general solution of $\frac{d}{dt}x = Ax$ where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Find e^{At}.

The eigenvalues are $\lambda=1$, $\lambda=2$ and λ_3.
The corresponding eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
and $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

then

$$\begin{aligned} X(t) &= e^{At} X_0 \\ &= e^{At}(X_{01}V_1 + X_{02}V_2 + X_{03}V_3) \\ &= X_{01} e^{At} v_1 + X_{02} e^{At} V_2 + X_{03} e^{At} V_3 \\ &= X_{01} e^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + X_{02} e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + X_{03} e^{3t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} X_{01} \\ X_{02} \\ X_{03} \end{bmatrix}. \end{aligned}$$

Hence $e^{At} = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$.

2. Find the general solution of $\frac{d}{dt}x = Ax$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Find e^{At}.

A has a single eigenvalue, $\lambda = 1$, of multiplicity 3. Since

$$[A - Id\lambda]^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

any set of linearly independent vectors generate the span of the generalized eigenvectors of A. We then can take

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

then

$$\begin{aligned} x(t) &= e^{At}(x_{01}v_1 + x_{02}v_2 + x_{03}v_3) \\ &= x_{01}e^{At}v_1 + x_{02}e^{At}v_2 + x_{03}e^{At}v_3 \\ &= x_{01}e^{t}\left(v_1 + t(A - \lambda Id)v_1 + \frac{t^2}{2}(A - \lambda Id)^2 v_1\right) \\ &\quad + x_{02}e^{t}\left(v_2 + t(A - \lambda Id)v_2 + \frac{t^2}{2}(A - \lambda Id)^2 v_2\right) \\ &\quad + x_{03}e^{t}\left(v_3 + t(A - \lambda Id)v_3 + \frac{t^2}{2}(A - \lambda Id)^2 v_3\right) \end{aligned}$$

Since $(A - I\lambda) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $(A - Id\lambda)^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we get

$$x(t) = x_{01}e^{t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_{02}e^{t}\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + x_{03}\begin{bmatrix} \frac{t^2}{2} + t \\ t \\ 1 \end{bmatrix}e^{t}$$

$$= e^{t}\begin{bmatrix} 1 & t & t + t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}$$

this implies that

$$e^{At} = e^{t}\begin{bmatrix} 1 & t & t + \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

3. Find the general solution of $\frac{d}{dt}x = Ax$ where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Provide a plot of the solutions.

the eigenvalues are the solutions of $\lambda^2 + 1 = 0$. Hence $\lambda = \pm i$.
the corresponding eigenvectors are $\begin{bmatrix} 1 \\ \mp i \end{bmatrix}$.
If $V = \begin{bmatrix} 1 \\ -i \end{bmatrix}$, then

$$X_0 = C_1 \operatorname{Re} V + C_2 \operatorname{Im} V$$

$$\Rightarrow \begin{bmatrix} X_{01} \\ X_{02} \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \begin{array}{l} C_1 = X_{01} \\ C_2 = -X_{02} \end{array}$$

Then

$$\begin{aligned} X(t) &= e^{At} X_0 \\ &= C_1 \operatorname{Re}(e^{At} V) + C_2 \operatorname{Im}(e^{At} V) \\ &= X_{01} \operatorname{Re}(e^{At} V) - X_{02} \operatorname{Im}(e^{At} V) \end{aligned}$$

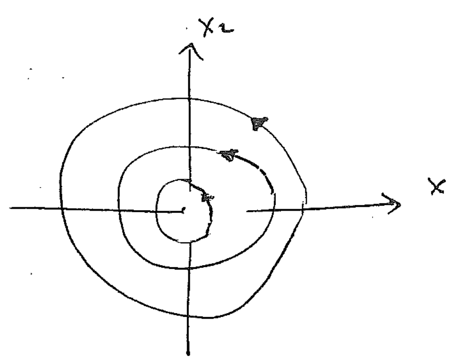
Since

$$e^{At} V = e^{it} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos t + i \sin t \\ -i \cos t + \sin t \end{bmatrix}$$

we get

$$\begin{aligned} X(t) &= X_{01} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} - X_{02} \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} \\ &= \cos t \begin{bmatrix} X_{01} \\ X_{02} \end{bmatrix} + \sin t \begin{bmatrix} -X_{02} \\ X_{01} \end{bmatrix} \end{aligned}$$

Hence the solutions lie on circles



4. Find e^{At} where $A = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$. *Hint: Solve the problem* $\frac{d}{dt}x = Ax$.

The eigenvalues are the solutions of $\lambda^2 - \text{trace}\,A\,\lambda + \det A = 0$. Since trace $A = \det A = 0$, the only eigenvalue is $\lambda = 0$ and it has multiplicity 2.

Since $(A - \lambda Id)^2 = A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, any set of linearly independent vectors generate the span of the generalized eigenvectors of $\lambda = 0$.

So, we take

$$V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then

$$\begin{aligned}
e^{At} x_0 &= e^{At}\left[x_{01} V_1 + x_{02} V_2\right] \\
&= x_{01} e^{At} V_1 + x_{02} e^{At} V_2 \\
&= x_{01}\left[V_1 + t\,(A - \lambda I) V_1\right] \\
&\quad + x_{02}\left[V_2 + t\,(A - \lambda I) V_2\right] \\
&= x_{01}\left[V_1 + tAV_1\right] + x_{02}\left[V_2 + tAV_2\right] \\
&= x_{01}\begin{bmatrix} 1 - 2t \\ 0 - t \end{bmatrix} + x_{02}\begin{bmatrix} 0 + 4t \\ 1 + 2t \end{bmatrix} \\
&= \begin{bmatrix} 1-2t & 4t \\ -t & 1+2t \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}
\end{aligned}$$

Hence

$$e^{At} = \begin{bmatrix} 1-2t & 4t \\ -t & 1+2t \end{bmatrix}$$

(A shorter proof: $e^{At} = Id + tA$ since $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.)

5. Find the solution of $\frac{d}{dt}x = Ax + f(t)$ and $x(0) = 0$, where $A = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$ and $f(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. *Hint: Find* e^{At} *by using the results in the previous problem.*

We have that

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-s)} f(s)\,ds.$$

Since $x(0) = 0$ and $f(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have

$$x(t) = \int_0^t e^{A(t-s)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} ds$$

By the previous result, $e^{A(t-s)} = \begin{bmatrix} 1-2(t-s) & 4(t-s) \\ -(t-s) & 1+2(t-s) \end{bmatrix}$,

and so

$$x(t) = \int_0^t \begin{bmatrix} 1-2(t-s) \\ -(t-s) \end{bmatrix} ds$$

$$= \begin{bmatrix} \int_0^t (1-2t+2s)\,ds \\ \int_0^t (-t+s)\,ds \end{bmatrix}$$

$$= \begin{bmatrix} t-2t^2+t^2 \\ -t^2+\frac{1}{2}t^2 \end{bmatrix}$$

$$= \begin{bmatrix} t-t^2 \\ -\frac{1}{2}t^2 \end{bmatrix}$$