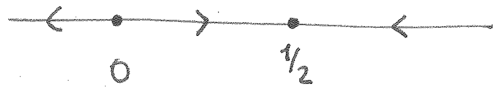
Fourth MATH 4512 mid-term exam: Dec. 16, 2015 NAME:
(Each problem is 4 points)

1. Consider the equation $\frac{d}{dt}x = 2x - 4x^2$. Without finding the solution $x(t)$, argue that, if $x(0) > 0$ then $x(t)$ tends to $1/2$ as t tends to infinity.

the phase diagram of the equation under consideration is



since $\frac{d}{dt}x = 4x\left(\frac{1}{2} - x\right)$. the equilibrium point $x=0$ is unstable and the equilibrium point $x=\frac{1}{2}$ is asymptotically stable. If $x(0) \in (0, 1/2)$ point $x(t)$ will increase as t increases and will tend to $1/2$ as t goes to infinity. If $x(0) \in (1/2, \infty)$, $x(t)$ will decrease as t increases and will tend to $1/2$ as t goes to infinity.

2. Find the equilibrium points of $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - x^3 - xy^2 \\ 2y - y^5 - yx^4 \end{bmatrix}$ and use linearization to determine their stability.

Set $f(x,y) = x - x^3 - xy^2$, $g(x,y) = 2y - y^5 - yx^4$.
the linearized system is

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \partial_x f(x,y) & \partial_y f(x,y) \\ \partial_x g(x,y) & \partial_y g(x,y) \end{bmatrix}\Bigg|_{(x_0,y_0)} \begin{bmatrix} u \\ v \end{bmatrix} = A\Big|_{(x_0,y_0)}\begin{bmatrix} u \\ v \end{bmatrix}$$

where (x_0, y_0) is an equilibrium point.
We have

$$A = \begin{bmatrix} 1 - 3x^2 - y^2 & -2xy \\ -4x^3y & 2 - 5y^4 - x^4 \end{bmatrix}$$

the equilibrium points are the solutions of

$$x(1 - x^2 - y^2) = 0$$
$$y(2 - y^4 - x^4) = 0$$

that is, $(0,0)$, $(0, \pm 2^{1/4})$, $(\pm 1, 0)$.

Hence

$$A(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow (0,0) \text{ is unstable}$$

$$A(0, \pm 2^{1/4}) = \begin{bmatrix} 1-\sqrt{2} & 0 \\ 0 & -8 \end{bmatrix} \Rightarrow (0, \pm 2^{1/4}) \text{ are asymptotically stable}$$

$$A(\pm 1, 0) = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow (\pm 1, 0) \text{ are saddles, and so, unstable.}$$

3. Find, and plot, the orbits of $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y/9 \\ -x/2 \end{bmatrix}$.

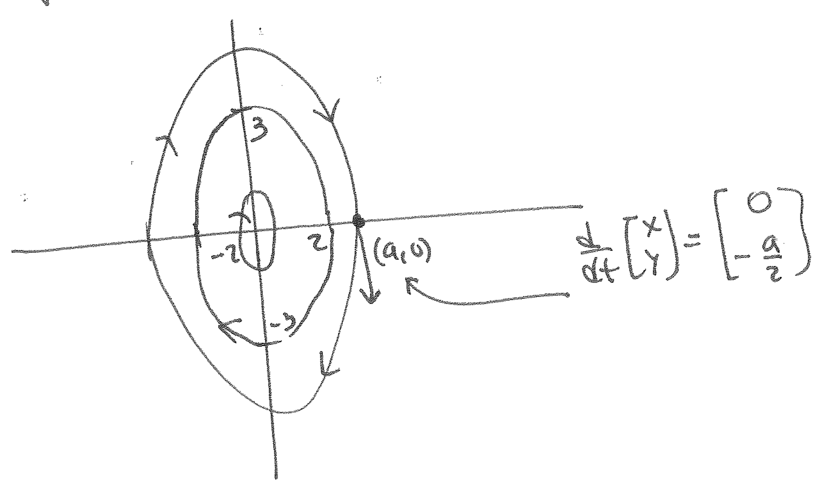
We have that $\frac{dy}{dx} = -\frac{x}{y}\frac{9}{4}$. This implies that the curves

$$\frac{y^2}{9} + \frac{x^2}{4} = \text{constant}$$

contain the orbits of the system. Since the only equilibrium point is $(x_0, y_0) = (0,0)$, the curves

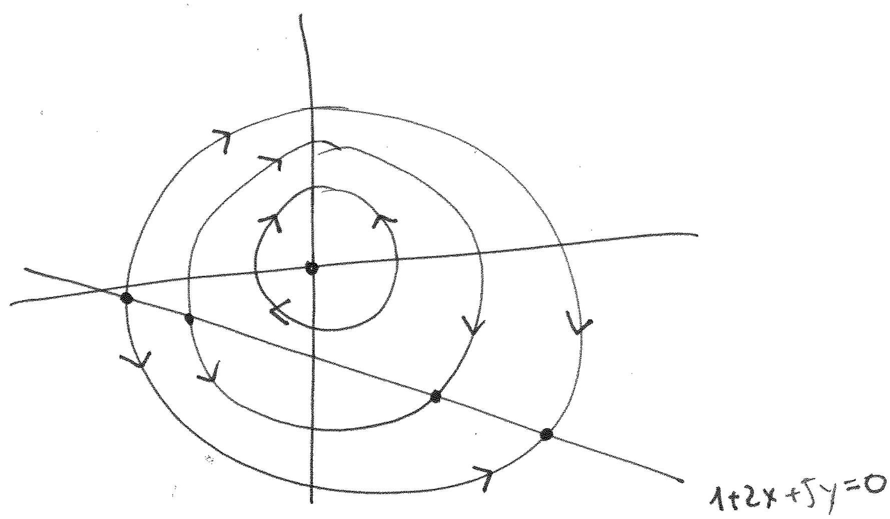
$$\frac{y^2}{9} + \frac{x^2}{4} = \text{constant}$$

contain periodic orbits. These ellipses are of the following form:



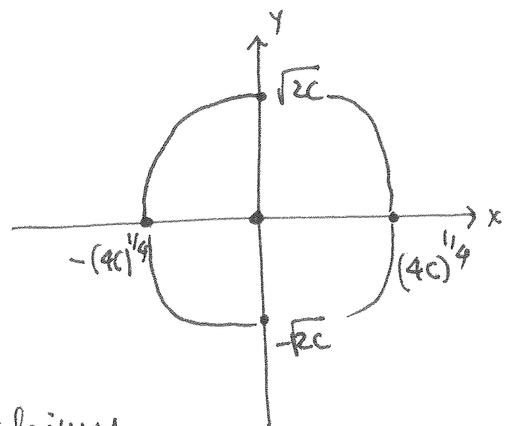
4. Find, and plot, the orbits of $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y\,(1+2x+5y) \\ -x\,(1+2x+5y) \end{bmatrix}$.

$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow y^2 + x^2 = c^2$ and so,
the orbits lie on the circles $x^2+y^2=c^2$.
Since the equilibrium points are
$(0,0)$ and (x_0,y_0) lying on the line $0=1+2x+5y$,
the phase portrait is the following:



5. Show that the orbits of $\frac{d^2}{dt^2}z + z^3 = 0$ are periodic.

Setting
$$x := z$$
$$y := \frac{dz}{dt},$$
the second-order equation becomes
$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x^3 \end{bmatrix}$$
the orbits lie on the solution of
$$\frac{dy}{dx} = \frac{-x^3}{y},$$
that is on the curves
$$\frac{1}{2}y^2 + \frac{1}{4}x^4 = C$$
Hence,
$$y = \pm\sqrt{2C - \frac{1}{2}x^4}$$



and we see that the orbits lie on a closed curve. For $C=0$, we have the only equilibrium point $(x,y)=(0,0)$. For $C>0$, there are no equilibrium points and so the orbits are periodic.