Math 4512 HW #4.

1.

$$y'' + \omega_0^2 y = 0 \quad \text{where} \quad \omega_0^2 = k/m.$$

$$k\Delta l = mg \Rightarrow k = mg/\Delta l$$

$$= 9.8 \times 1/(49/320)$$

$$= 64$$

The general solution of $y'' + \omega_0^2 y = 0$ is

$$y(t) = a\cos\omega_0 t + b\sin\omega_0 t$$

$$= R\cos(\omega_0 t - \delta) \quad \text{by lemma 1}$$

where $R = \sqrt{a^2+b^2}$ and $\delta = \tan^{-1} b/a$.

$y(0) = 1/4$ and $y'(0) = 0 \Rightarrow a = \frac{1}{4}$, $b = 0$

Hence, $y(t) = \sqrt{(\frac{1}{4})^2 + 0}\,\cos(\sqrt{64}\,t - 0)$

$$= \frac{1}{4}\cos(8t)$$

Frequency $\omega_0 = 8$

Period $T_0 = 2\pi/\omega_0 = \pi/4$

Amplitude $R = 1/4$

5.

Let $y(t)$ be the position of the mass away from the equibrium position at t

$m=1$, $k=1$, $c=2$

Hence,

$$y''+2y'+y=0 \; ; \; y(0)=\frac{1}{4}, \; y'(0)=-1$$

The characteristic equation is

$$r^2+2r+1=0$$

$$\Rightarrow \; r_1=r_2=-1$$

The general solution of ODE is

$$y(t)=C_1e^{-t}+C_2te^{-t}$$

$y(0)=\frac{1}{4}$, $y'(0)=-1 \Rightarrow C_1=\frac{1}{4}$, $C_2=-\frac{3}{4}$

Hence,

$$y(t)=\frac{1}{4}e^{-t}-\frac{3}{4}te^{-t}=\frac{1}{4}e^{-t}(1-3t)$$

At $t=\frac{1}{3}$, $y(\frac{1}{3})=0$ and $t=\frac{1}{3}$ is the only solution that satisfies $y(t)=0$. So this means the mass will overshoot its equilibrium position once. On the other hand, $y(t)\to 0$ as $t\to\infty$, which means the mass will creep back to equilibrium as $t\to\infty$.

5.

$$y'' - 5y' + 4y = e^{2t}; \quad y(0) = 1, \; y'(0) = -1$$

Solution:

Let $Y(s) = \mathcal{L}\{y(t)\}$. Taking Laplace transform of both sides of the differential equation gives

$$s^2 Y(s) - s + 1 - 5[sY(s) - 1] + 4Y(s) = \frac{1}{s-2}$$

(Notice that $\mathcal{L}\{y'(t)\}$ and $\mathcal{L}\{y''(t)\}$ are computed by lemmas 3 and 4)

and implies that

$$Y(s) = \frac{(s-6)(s-2)+1}{(s-2)(s-1)(s-4)}$$

$$= \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-4}$$

$\Rightarrow A(s-2)(s-4) + B(s-1)(s-4) + C(s-1)(s-2) = (s-2)(s-6) + 1$

$s = 2 \Rightarrow B = -\frac{1}{2}$, $s = 4 \Rightarrow C = -\frac{1}{2}$, $s = 1 \Rightarrow A = 2$

Thus, $Y(s) = 2\frac{1}{s-1} - \frac{1}{2}\frac{1}{s-2} - \frac{1}{2}\frac{1}{s-4}$

Therefore, $Y(s) = \mathcal{L}\{2e^t - \frac{1}{2}e^{2t} - \frac{1}{2}e^{4t}\}$

so that

$$y(t) = 2e^t - \frac{1}{2}e^{2t} - \frac{1}{2}e^{4t}$$

There is also an alternative way to do this.

$$y(t) = Ae^{2t}$$

is a very good guess of the particular solution.

Thus, $y'(t) = 2Ae^{2t}$ and $y''(t) = 4Ae^{2t}$

$$4Ae^{2t} - 10Ae^{2t} + 4Ae^{2t} = e^{2t}$$

$$\Rightarrow A = -\frac{1}{2} \Rightarrow y_p = -\frac{1}{2}e^{2t}$$

To solve the corresponding homogeneous equation

$$y'' - 5y' + 4y = 0$$

we note that

$$r^2 - 5r + 4 = 0 \Rightarrow r = 1 \text{ and } 4$$

The general solution of $y'' - 5y' + 4y = 0$ is

$$y_h = C_1 e^t + C_2 e^{4t}$$

$$\Rightarrow y(t) = y_p + y_h = -\frac{1}{2}e^{2t} + C_1 e^t + C_2 e^{4t}$$

$y(0) = 1$ and $y'(0) = -1 \Rightarrow y(t) = 2e^t - \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t}$

$\blacksquare$

Notice: I do not recommend this unless you have trouble with Laplace transform.

21.

$$y'' - 2y' + y = te^t; \quad y(0) = 0, \; y'(0) = 0$$

The Laplace transform of e^t is $1/(s-1)$. Hence,

$$\mathcal{L}\{te^t\} = -\frac{d}{ds}\frac{1}{s-1} = \frac{1}{(s-1)^2}$$

by property 1.

Then, we apply the same method with last problem

$$s^2Y(s) - 2sY(s) + Y(s) = \frac{1}{(s-1)^2}$$

$$\Rightarrow Y(s) = \frac{1}{(s-1)^4}$$

We recognize that

$$\frac{1}{(s-1)^4} = \frac{d^3}{ds^3}\left(-\frac{1}{6}\right)\frac{1}{s-1}$$

$$= \mathcal{L}\{\tfrac{1}{6}t^3e^t\}$$ by using property 1 three times.

Hence, $y(t) = \frac{1}{6}t^3e^t$

$\blacksquare$