MATH 4512 final exam: Dec. 21, 2016 NAME:
(Each problem is 4 points)

1. Find the general solution of $\frac{d^2}{dt^2}x - 2\frac{d}{dt}x + x = 0$.

Setting $Y := \frac{d}{dt}X$, we rewrite the original equation as follows:

$$\frac{d}{dt}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$$

Thus, the general solution is

$$\begin{bmatrix} X \\ Y \end{bmatrix}(t) = e^{At}\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$.

The characteristic polynomial is $\det(A - \lambda Id) = (\lambda-1)^2$.

Since $(A - \lambda Id)\Big|_{\lambda=1} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$,

we have that $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} V_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow V_1 = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(we set $a=1$), and $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} V_2 = V_1 \Rightarrow V_2 = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Hence

$$A = [V_1 V_2]\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}[V_1 V_2]^{-1}$$

$$\Rightarrow e^{At} = [V_1 V_2]\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}(V_1 V_2)^{-1}$$

$$= e^t [V_1 V_2]\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}[V_1 V_2]^{-1}$$

$$= e^t \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= e^t \begin{bmatrix} 1 & t \\ 1 & 1+t \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= e^t \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}.$$

The general solution is then

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = e^t\begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}.$$

In particular,

$$X(t) = (1-t)e^t X_0 + t Y_0 e^t.$$

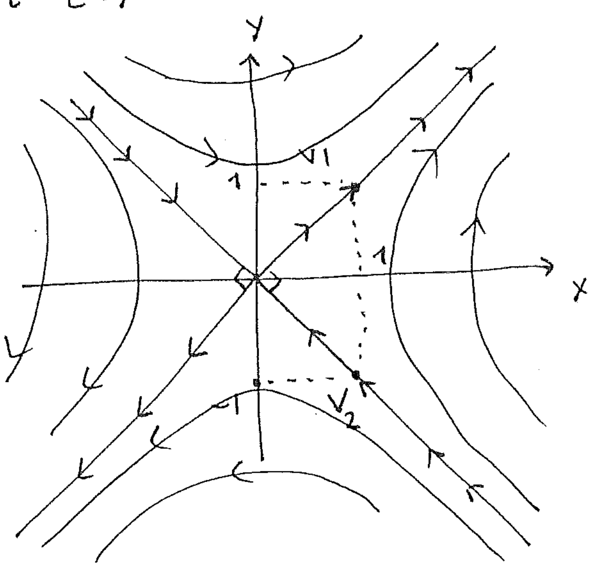
2. Plot the phase portrait of the equation $\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2y\\2x\end{bmatrix}$.

$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0 & 2\\2 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$.

the characteristic polynomial of $A = \begin{bmatrix}0 & 2\\2 & 0\end{bmatrix}$ is $\det(A - \lambda Id) = \lambda^2 - 4$, and so the eigenvalues of A are $\lambda_1 = +2$ and $\lambda_2 = -2$. Let us find the eigenvectors. Since

$$(A - \lambda_1 Id) = \begin{bmatrix}-2 & 2\\2 & -2\end{bmatrix}$$
$$(A - \lambda_2 Id) = \begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}$$

we can take the eigenvector associated to λ_1 as $V_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and the one associated to λ_2 as $V_2 = \begin{bmatrix}1\\-1\end{bmatrix}$. We can now get the phase portrait.



3. Find and plot the orbits of $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y\,\cos(x^2+y^2) \\ -x\,\cos(x^2+y^2) \end{bmatrix}$.

The orbits satisfy the equation

$$\frac{d}{dx}y = \frac{(-x\ \cos(x^2+y^2))}{(y\ \cos(x^2+y^2))} = -\frac{x}{y}$$

$$\Rightarrow \frac{d}{dx}(y^2+x^2) = 0$$

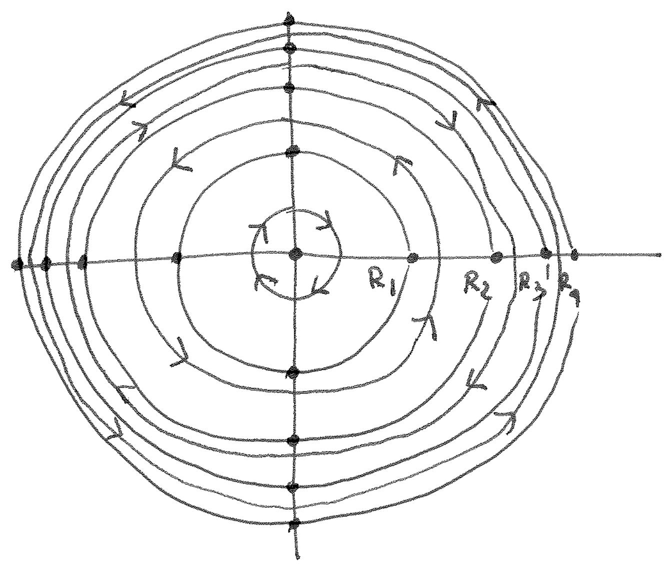
Hence, the orbits are the circles $x^2+y^2=R^2$. Each of the points on the circle with radius R are equilibrium points provided $\cos R^2 = 0$, that is, provided $R^2 = \frac{\pi}{2} + n\pi$, for all $n \in \mathbb{Z}$, or $R=0$. Finally at $(x,y)=(R,0)$, $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -R\cos R^2 \end{bmatrix}$. the sign of $R\cos R^2$ is negative when

$$\frac{\pi}{2} + 2m\pi < R^2 < \frac{\pi}{2} + (2m+1)\pi, \quad m \in \mathbb{Z}.$$

and positive if

$$\frac{\pi}{2} + (2m-1)\pi < R^2 < \frac{\pi}{2} + 2m\pi, \quad m \in \mathbb{Z}.$$

We can now plot the orbits:



$$R_i = \sqrt{\frac{\pi}{2}}\sqrt{(2i-1)}$$

4. Find the equilibrium points of $\frac{d}{dt}\begin{bmatrix} S \\ N \end{bmatrix} = \begin{bmatrix} S\,(1 - N/2) \\ N\,(1 - N - S) \end{bmatrix}$. Linearize the equations around each equilibrium point and draw the corresponding phase portrait. Then deduce the stability properties of the original equilibrium points from those of the corresponding linearization.

The equilibrium points are those (S_0, N_0) such that

$$S_0\left(1 - \frac{N_0}{2}\right) = 0 \quad \text{and} \quad N_0(1 - N_0 - S_0) = 0.$$

Hence, we have that

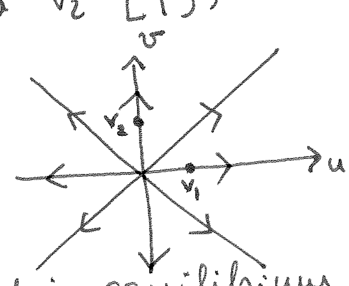
$$(S_0, N_0) \in \{(0,0),\ (0,1),\ (-1,2)\}.$$

Linearization around the equilibrium points gives

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 1 - N_0/2 & -S_0/2 \\ -N_0 & 1 - 2N_0 - S_0 \end{bmatrix}}_{A_0.}\begin{bmatrix} u \\ v \end{bmatrix}$$

- For $(S_0, N_0) = (0,0)$:

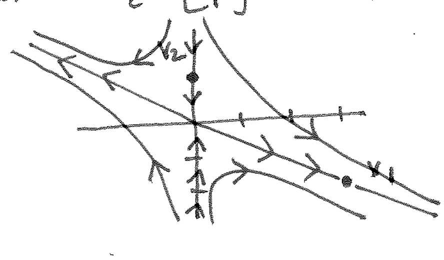
$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\lambda = 1$ is a repeated eigenvalue with algebraic and geometric multiplicities equal to 2. We can take its eigenvectors to be $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so that the phase portrait is



the corresponding equilibrium point for the original problem is unstable

- For $(S_0, N_0) = (0,1)$:

$A_0 = \begin{bmatrix} 1/2 & 0 \\ -1 & -1 \end{bmatrix}$ and $\lambda_1 = 1/2$, $\lambda_2 = -1$. Since $(A_0 - \lambda_1 Id) = \begin{bmatrix} 0 & 0 \\ -1 & -3/2 \end{bmatrix}$, we can take $v_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Since $(A_0 - \lambda_2 Id) = \begin{bmatrix} 3/2 & 0 \\ -1 & 0 \end{bmatrix}$, we can take $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The phase portrait is:



the original equilibrium point is unstable.

For $(S_0, N_0) = (-1, 2)$:

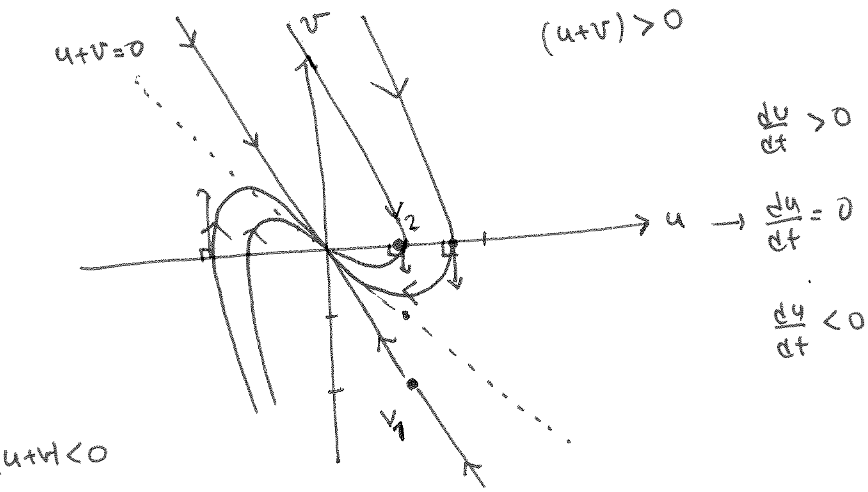
$A_0 = \begin{bmatrix} 0 & 1/2 \\ -2 & -2 \end{bmatrix}$, the characteristic polynomial is $\lambda^2 + 2\lambda + 1 = (\lambda+1)^2$. Hence $\lambda = -1$ is a repeated eigenvalue. Since $(A_0 - \lambda Id) = \begin{bmatrix} 1 & 1/2 \\ -2 & -1 \end{bmatrix}$ we can take $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ as the only eigenvector (up to a constant.) If v_2 is to solve $(A_0\ \lambda Id) v_2 = v_1$, we can take $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Finally, note that $\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v/2 \\ -(u+v) \end{bmatrix}$ so that

$\frac{du}{dt} > 0$ if $v > 0$ and $\frac{du}{dt} < 0$ if $v < 0$

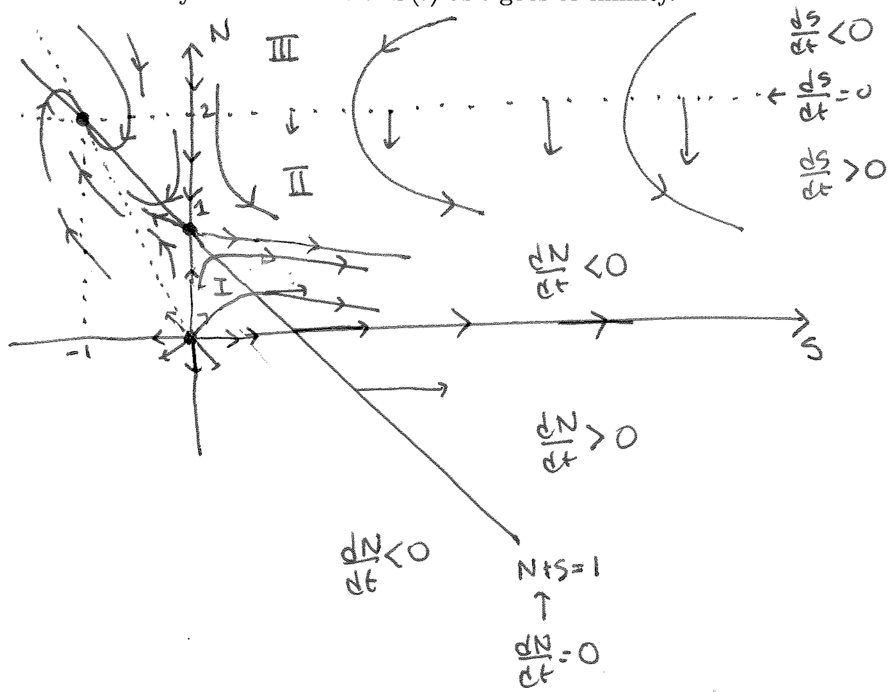
$\frac{dv}{dt} > 0$ if $u+v < 0$ and $\frac{dv}{dt} < 0$ if $u+v > 0$.

We can now draw the phase portrait.



The original equilibrium point is stable.

5. Use the result of the preceding problem to sketch the phase portrait of $\frac{d}{dt}\begin{bmatrix} S \\ N \end{bmatrix} = \begin{bmatrix} S\,(1-N/2) \\ N\,(1-N-S) \end{bmatrix}$. Assuming that $S(0) > 0$ and $N(0) > 0$, argue $S(t) > 0$ and $N(t) > 0$ for all $t > 0$. Then show that $N(t)$, as t increases, will eventually be less than 2. Finally find the limit of $S(t)$ as t goes to infinity.



Region I : $N>0,\ S>0,\ N+S<1$
Region III : $N>2$
Region II : $N>0,\ S>0,\ N<2,\ N+S>1$

We see that on the line $S=0$, $\frac{d}{dt}\begin{bmatrix} S \\ N \end{bmatrix} = \begin{bmatrix} 0 \\ N(1-N) \end{bmatrix}$. This means that no orbit starting with $N_{(0)}>0$, $S_{(0)}>0$ can escape through the line $S=0$.
Similarly, no orbit starting with $N(0)>0$, $S(0)>0$ can escape through the line $N=0$ since $\frac{d}{dt}\begin{bmatrix} S \\ N \end{bmatrix} = \begin{bmatrix} S \\ 0 \end{bmatrix}$.
If $N(0)>2$ and $S(0)>0$, we have that $\frac{dN}{dt}<0$. Since there are no equilibrium points in Region III, and $\frac{dS}{dt}<0$, the orbit has to touch the line $N=2$, $S>0$. There, $\frac{dN}{dt}<0$ and $\frac{dS}{dt}=0$ so that the orbit must enter the region II. If $(S(0), N(0))$ are inside region I, a similar argument shows that the corresponding orbit leaves region I into region II. Since $\frac{dS}{dt}>0$ in region II, $\lim_{t\to\infty} S(t) = \infty$.