Second MATH 4512 mid-term exam: Oct. 28, 2016 NAME:

(Each problem is 5 points)

1. Find the Laplace transform of $e^{bt} \cos at$.

To find the Laplace transform of $y(t) = e^{bt} \cos at$, we find the Laplace transform of $z(t) = e^{(b+ia)t}$ and then we take its real part.

So

$$\mathcal{L}\{z(t)\} = \mathcal{L}\{e^{(b+ia)t}\}$$

$$= \frac{1}{s-(b+ia)}$$

$$= \frac{1}{(s-b)+ia}$$

$$= \frac{(s-b)-ia}{(s-b)^2+a^2}$$

and hence

$$\mathcal{L}\{y(t)\} = \mathrm{Re}\, \mathcal{L}\{z(t)\}$$

$$= \frac{(s-b)}{(s-b)^2+a^2}.$$

2. Find the function whose Laplace transform is $s^2/(s^2+4)^2$.

We have :

$$\frac{s^2}{(s^2+4)^2} = s \cdot \frac{s}{(s^2+4)^2}$$

$$= s \cdot \frac{d}{ds}\left(\frac{1}{s^2+4}\right)\left(-\frac{1}{2}\right)$$

$$= s \cdot \frac{d}{ds} \mathcal{L}\left\{\frac{\sin 2t}{2}\right\} \cdot \left(-\frac{1}{2}\right)$$

$$= -\frac{s}{4} \frac{d}{ds} \mathcal{L}\{\sin 2t\}$$

$$= -\frac{s}{4} \mathcal{L}\{-t \sin 2t\}$$

$$= \frac{s}{4} \mathcal{L}\{t \sin 2t\}$$

$$= \frac{1}{4} \mathcal{L}\left\{\frac{d}{dt}(t \sin 2t)\right\}$$

$$= \mathcal{L}\left\{\frac{1}{4}(\sin 2t + 2t \cos 2t)\right\}$$

and so, the function we seek is

$$y(t) = \frac{1}{4} \sin 2t + \frac{t}{2} \cos 2t \ .$$

3. Using the Laplace transform, find the solution of the initial-value problem $\frac{d^2}{dt^2}y + 4y = -\frac{1}{2}\sin 2t$, $y(0) = 0$ and $y'(0) = \frac{1}{4}$.

If we set $Y = \mathcal{L}\{y(t)\}$, we get that

$$\left(s^2 Y - y'(0) - sy(0)\right) + 4Y = -\frac{1}{2}\mathcal{L}\{\sin 2t\}$$

$$\Rightarrow \quad (s^2+4)Y - \frac{1}{4} = -\frac{1}{2}\cdot\frac{2}{s^2+4}$$

$$\Rightarrow \quad Y = \left[-\frac{1}{s^2+4} + \frac{1}{4}\right]\cdot\left[\frac{1}{s^2+4}\right]$$

$$= \frac{1}{4}\frac{s^2}{(s^2+4)^2}$$

$$= \frac{1}{4}\mathcal{L}\left\{\frac{1}{4}\sin 2t + \frac{t}{2}\cos 2t\right\} \quad \text{by problem 2.}$$

hence

$$y(t) = \frac{1}{16}\sin 2t + \frac{t}{8}\cos 2t.$$

4. A spring of unit mass is immersed in a medium with a drag coefficient equal to 2. The spring is chosen in such a way that its movement is *critically damped.* Using the Laplace transform, find the position of the spring as a function of time for the case in which the initial displacement $y(0) := y_0$ is arbitrary and $y'(0) = 0$. Show that the spring goes back to its equilibrium position without oscillating a single time.

The equation governing the movement of the spring is

$$m \frac{d^2}{dt^2} y + c \frac{d}{dt} y + k y = 0$$
$$y(0) = y_0$$
$$y'(0) = 0$$

Since the movement of the spring is "critically damped", we must have that $c^2 = 4km$. Since $m=1$ and $c=2$, we have that $k=1$.

If $Y = \mathcal{L}\{y(t)\}$, we get that

$$\left(s^2 Y - y'(0) - s y(0)\right) + 2\left(sY - y(0)\right) + kY = 0$$
$$(s^2+2s+1) Y = y(0)\,(s+2)$$
$$\Rightarrow \quad Y = y_0 \frac{s+2}{(s+1)^2}$$
$$= y_0 \left[\frac{(s+1)+1}{(s+1)^2} \right]$$
$$= y_0 \left[\frac{1}{s+1} - \frac{d}{ds}\left(\frac{1}{s+1}\right) \right]$$
$$= y_0 \left[\mathcal{L}\{e^{-t}\} + \mathcal{L}\{t e^{-t}\} \right]$$

and so

$$y(t) = y_0 (1+t) e^{-t}.$$

Clearly, $\lim_{t \to \infty} y(t) = 0$ and so the spring goes back to its equilibrium position. Moreover, since $y'(t) = y_0(-t)e^{-t}$, its velocity is never zero and so the spring does not oscillate a single time.