Third MATH 4512 mid-term exam: Nov. 28, 2016 NAME:

(Each problem is 5 points)

1. Find the general solution of $\frac{d}{dt}x = Ax$ where $A = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}$. Find e^{At}.

The characteristic polynomial is $p(\lambda) = \det(A - \lambda Id)$

$= \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4)$.

An eigenvector v_1 associated to $\lambda_1 = 3$ satisfies

$\begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} a \\ a \end{bmatrix}$. For $a = 1$, $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

An eigenvector v_2 associated to $\lambda_2 = 4$ satisfies

$\begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 3a \\ 2a \end{bmatrix}$. For $a = 1$, $v_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

If $x_0 = c_1 v_1 + c_2 v_2$ than the general solution is

$$\begin{aligned} x(t) &= e^{At}(c_1 v_1 + c_2 v_2) \\ &= c_1 e^{At} v_1 + c_2 e^{At} v_2 \\ &= c_1 e^{3t} v_1 + c_2 e^{4t} v_2 \\ &= c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \end{aligned}$$

to find e^{At}, we write that

$$\begin{aligned} x(t) &= \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}^{-1} x_0 \end{aligned}$$

and so

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}^{-1}. \\ &= \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} e^{3t} & 3e^{4t} \\ e^{3t} & 2e^{4t} \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -2e^{3t} + 3e^{4t} & 3e^{3t} - 3e^{4t} \\ -2e^{3t} + 2e^{4t} & 3e^{3t} - 2e^{4t} \end{bmatrix}. \end{aligned}$$

2. Find the general solution of $\frac{d}{dt}x = Ax$ where $A = \begin{bmatrix}-3 & 2\\ -1 & -1\end{bmatrix}$. Find e^{At}.

the characteristic polynomial is

$$p(\lambda) = \lambda^2 + 4\lambda + 5 \quad \Rightarrow \quad \lambda = \frac{1}{2}\left(-4 \pm \sqrt{16-20}\right) = -2 \pm i$$

An eigenvector associated to $\lambda_1 = -2+i$ satisfies

$$\begin{bmatrix}-1-i & 2\\ -1 & 1-i\end{bmatrix} v_1 = \begin{bmatrix}0\\ 0\end{bmatrix} \Rightarrow v_1 = \begin{bmatrix}1-i\\ 1\end{bmatrix} b, \text{ For } b=1,\ v_1 = \begin{bmatrix}1-i\\ 1\end{bmatrix}.$$

Since $\lambda_2 = -2-i = \bar{\lambda}_1$, we can take the eigenvector v_2 associated to λ_2 as $v_2 := \bar{v}_1$.

Then, if $x_0 = c_1 v_1 + c_2 v_2$,

$$\begin{aligned}
x(t) &= e^{At} x_0 \\
&= c_1 e^{At} v_1 + c_2 e^{At} v_2 \\
&= c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 \\
&= c_1 e^{-2t} \cdot e^{it} \begin{bmatrix}1-i\\ 1\end{bmatrix} + c_2 e^{-2t} e^{-it} \begin{bmatrix}1+i\\ 1\end{bmatrix} \\
&= e^{-2t} \begin{bmatrix}1-i & 1+i\\ 1 & 1\end{bmatrix} \begin{bmatrix}e^{it} & 0\\ 0 & e^{-it}\end{bmatrix} \begin{bmatrix}1-i & 1+i\\ 1 & 1\end{bmatrix}^{-1} x_0 \\
&= e^{-2t} \begin{bmatrix}1-i & 1+i\\ 1 & 1\end{bmatrix} \begin{bmatrix}e^{it} & 0\\ 0 & e^{-it}\end{bmatrix} \frac{i}{2} \begin{bmatrix}1 & -1-i\\ -1 & 1-i\end{bmatrix} x_0 \\
&= e^{-2t} \begin{bmatrix}(1-i)e^{it} & (1+i)e^{-it}\\ e^{it} & e^{-it}\end{bmatrix} \begin{bmatrix}i/2 & -i/2 + 1/2\\ -i/2 & i/2 + 1/2\end{bmatrix} x_0 \\
&= e^{-2t} \begin{bmatrix}\frac{i}{2}(1-i)e^{it} - \frac{i}{2}(1+i)e^{-it} & (1-i)\left(\frac{1-i}{2}\right)e^{it} + (1+i)\left(\frac{1+i}{2}\right)e^{-it}\\ \frac{i}{2} \cdot e^{it} - \frac{i}{2} \cdot e^{-it} & \left(\frac{1-i}{2}\right)e^{it} + \left(\frac{1+i}{2}\right)e^{-it}\end{bmatrix} x_0 \\
&= e^{-2t} \begin{bmatrix}\mathrm{Re}\left(i(1-i)e^{it}\right) & \mathrm{Re}\left((1-i)(1-i)e^{it}\right)\\ \mathrm{Re}\left(ie^{it}\right) & \mathrm{Re}\left((1-i)e^{it}\right)\end{bmatrix} x_0 \\
&= e^{-2t} \begin{bmatrix}\cos t - \sin t & 2\sin t\\ -\sin t & \cos t + \sin t\end{bmatrix} x_0
\end{aligned}$$

$$\Rightarrow e^{At} = e^{-2t} \begin{bmatrix}\cos t - \sin t & 2\sin t\\ \sin t & \cos t + \sin t\end{bmatrix}.$$

3. Find the general solution of $\frac{d}{dt}x = Ax$ where $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$. Find e^{At}.

the characteristic polynomial is

$p(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2.$

Now, $\quad (A - \lambda Id)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

and so any two linearly independent vectors span the corresponding space of generalized eigenvectors. We take

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

then

$$\begin{aligned}
X(t) &= e^{At} x_0 \\
&= e^{At}(c_1 v_1 + c_2 v_2) \\
&= c_1 e^{At} v_1 + c_2 e^{At} v_2 \\
&= c_1 \left\{ e^{2t} (Id + t(A - 2Id)) v_1 \right\} \\
&\quad + c_2 \left\{ e^{2t} (Id + t(A - 2Id)) v_1 \right\} \\
&= e^{2t} (Id + t(A - 2Id)) (c_1 v_1 + c_2 v_2) \\
&= e^{2t} (Id + t(A - 2Id)) x_0 \\
&= e^{2t} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \right) x_0 \\
&= e^{2t} \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix} x_0
\end{aligned}$$

$$\Rightarrow e^{At} = e^{2t} \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix}.$$

4. Find the solution of $\frac{d}{dt}x = Ax + f, x(0) = x_0$ where $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}, f(t) = \begin{bmatrix} -t^2 \\ 2t \end{bmatrix}$ and $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. *Hint: Use e^{At} obtained in the previous problem.*

In this case, we have that

$$x(t) = \int_0^t e^{+A(t-s)} f(s)\, ds$$

By the previous result,

$$e^{A(t-s)} = e^{+2t} e^{-2s} \begin{bmatrix} 1-t+s & -t+s \\ t-s & 1+t-s \end{bmatrix},$$

and so

$$x(t) = \int_0^t e^{2t} e^{-2s} \begin{bmatrix} (1-t+s)(-s^2) + (-t+s)(2s) \\ (t-s)(-s^2) + (1+t-s)(2s) \end{bmatrix} ds$$

$$= e^{2t} \left\{ \begin{bmatrix} -2t \\ 2(1+t) \end{bmatrix} \int_0^t e^{-2s} \cdot s\, ds + \begin{bmatrix} 1+t \\ -(2+t) \end{bmatrix} \int_0^t e^{-2s} \cdot s^2\, ds \right.$$

$$\left. + \begin{bmatrix} -1 \\ +1 \end{bmatrix} \int_0^t e^{-2s} \cdot s^3\, ds \right\}.$$

(The simplified solution is:

$$x(t) = \begin{bmatrix} 3/4 \\ -1/4 \end{bmatrix} t^2 + \begin{bmatrix} 1/2 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1/8 \\ -3/8 \end{bmatrix} (1 - e^{2t}) + \begin{bmatrix} -1/4 \\ 1/4 \end{bmatrix} t e^{2t}.)$$