By popular demand, here is a solution of problem 13 of section §4.7.

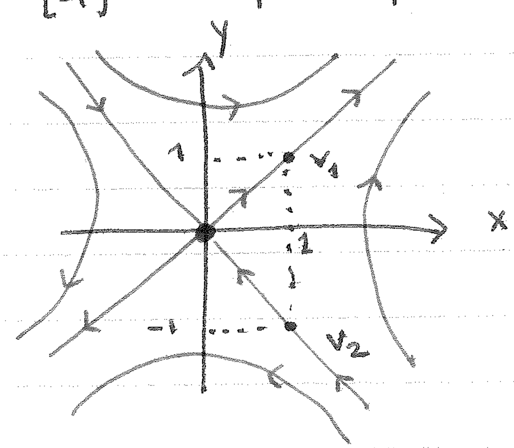
a) If we linearize the system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x + 2x^3 \end{bmatrix}$$

around the equilibrium point $(x,y) = (0,0)$, we get

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix}$$

The characteristic polynomial of A is $p(\lambda) = \det(A - \lambda \mathrm{Id}) = \lambda^2 - 1$ and so the eigenvalues are $\lambda_1 = +1$ and $\lambda_2 = -1$. Since $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, an eigenvector associated to λ_1 is $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and an eigenvector associated to λ_2 is $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The phase portrait is the following:



The equilibrium point $(0,0)$ is clearly a saddle.

b) To find the orbits, we solve

$$\frac{dy}{dx} = \frac{x+2x^3}{y}$$

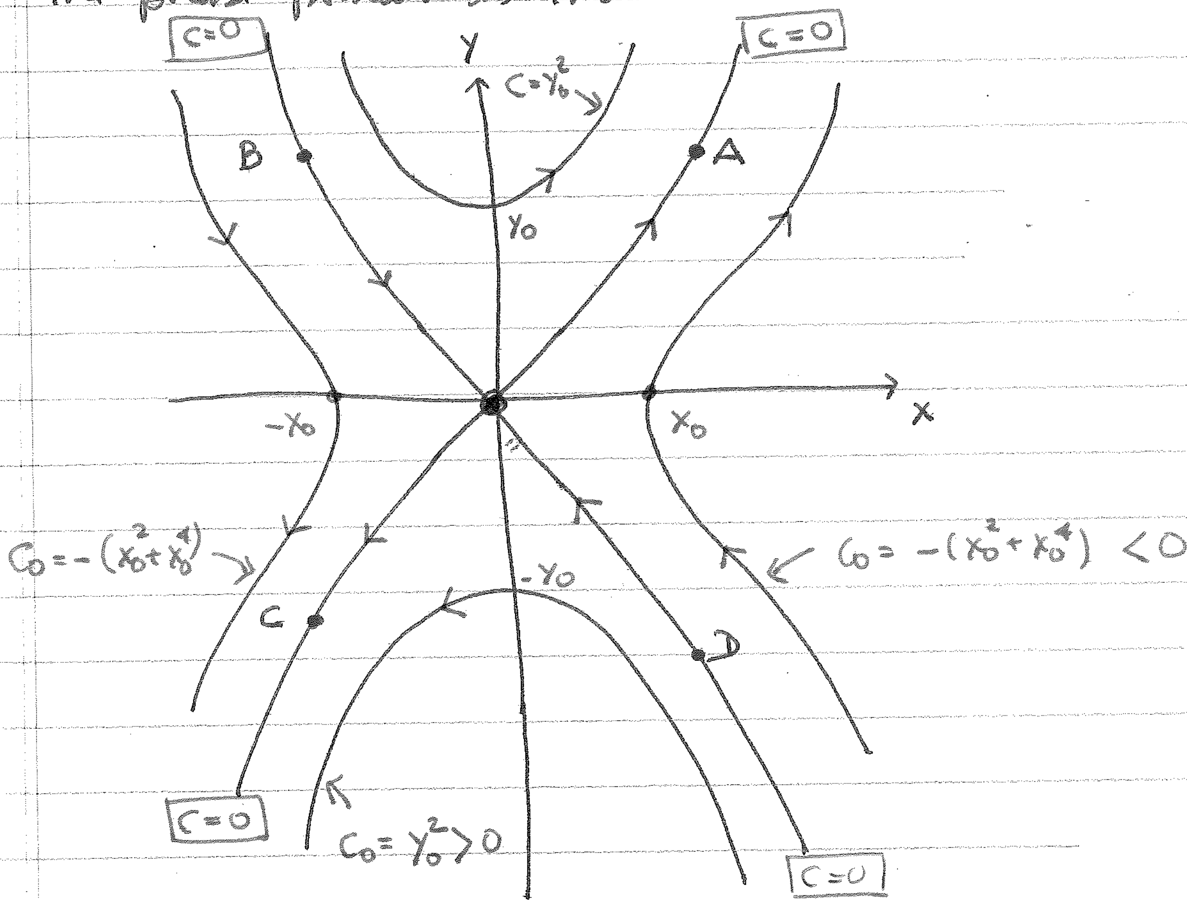
$$\Rightarrow \quad y\frac{dy}{dx} = x + 2x^3$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{1}{2}y^2\right) = \frac{d}{dx}\left(\frac{1}{2}x^2 + \frac{1}{2}x^4\right)$$

$$\Rightarrow \quad y^2 = x^2 + x^4 + c$$

$$\Rightarrow \quad y = \pm\sqrt{x^2+x^4+c}\,.$$

The phase portrait is then



c) the orbits containing the points B and D go to zero as $t \to \infty$. Those containing the points C and A go to zero as $t \to -\infty$.