Second MATH 4512 mid-term exam: Oct. 27, 2017 NAME:

(Each problem is 5 points)

1. Find the Laplace transform of $t^2 e^{bt}$.

We have

$$\mathcal{L}\{t^2 e^{bt}\} = \frac{d}{ds}\mathcal{L}\{-t e^{bt}\} \quad \text{since} \quad \frac{d}{ds}\mathcal{L}\{f(t)\} = \mathcal{L}\{-t f(t)\}$$

$$= \frac{d^2}{ds^2}\mathcal{L}\{e^{bt}\} \quad \text{for the same reason}$$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s-b}\right)$$

$$= \frac{2}{(s-b)^3}.$$

2. Find the function whose Laplace transform is $s^2/(s^2 - a^2)$.

$$\frac{s^2}{s^2-a^2} = 1 + \frac{a^2}{s^2-a^2}$$

$$= 1 + a^2\left[\frac{1/2a}{s-a} - \frac{1/2a}{s+a}\right]$$

$$= \mathcal{L}\{\delta_0(t)\} + \frac{a}{2}\mathcal{L}\{e^{at} - e^{-at}\}$$

$$= \mathcal{L}\{\delta_0(t) + \frac{a}{2}(e^{at} - e^{-at})\}$$

So, the Laplace transform of $\delta_0(t) + \frac{a}{2}(e^{at} - e^{-at})$ is $\frac{s^2}{s^2-a^2}$.

3. Using the Laplace transform, find the solution of the initial-value problem $\frac{d}{dt}y + 4y = e^{4t}$, $y(0) = 1$.

$$\mathcal{L}\left\{\frac{d}{dt}y\right\} + 4\mathcal{L}\{y\} = \frac{1}{s-4}$$

$$\Rightarrow (s+4)\,\mathcal{L}\{y(t)\} - 1 = \frac{1}{s-4}$$

$$\Rightarrow \mathcal{L}\{y(t)\} = \frac{1}{s+4} + \frac{1}{s+4}\cdot\frac{1}{s-4}$$

$$= \mathcal{L}\{e^{-4t}\} + \mathcal{L}\{e^{-4t}\}\cdot\mathcal{L}\{e^{4t}\}$$

$$\Rightarrow y(t) = e^{-4t} + \int_0^t e^{-4u}\, e^{4(t-u)}\, du$$

$$= e^{-4t} + e^{4t}\int_0^t e^{-8u}\, du$$

$$= e^{-4t} + e^{4t}\left(\frac{e^{-8t}-1}{-8}\right)$$

$$= e^{-4t} + \frac{1}{8}e^{4t} - \frac{1}{8}e^{-4t}$$

$$= \frac{1}{8}e^{4t} + \frac{7}{8}e^{-4t}$$

4. Using the Laplace transform, find the solution of the initial-value problem $\frac{d^2}{dt^2}y + 4y = c\,\delta_\pi$, $y(0) = 0$ and $y'(0) = 1$. Find the value of the parameter c for which $y(t) = 0$ for all $t > \pi$.

$$s^2 \mathcal{L}\{y(t)\} - (s\,y(0) + y'(0)) + 4\,\mathcal{L}\{y(t)\} = c\,\mathcal{L}\{\delta_\pi\}$$

$$\mathcal{L}\{y(t)\} = \frac{1}{s^2+4} + c\,\frac{1}{s^2+4}\,\mathcal{L}\{\delta_\pi\}$$

$$= \mathcal{L}\left\{\frac{\sin 2t}{2}\right\} + c\,\mathcal{L}\left\{\frac{\sin 2t}{2}\right\}\mathcal{L}\{\delta_\pi\}$$

$$\Rightarrow \quad y(t) = \frac{\sin 2t}{2} + c\int_0^t \delta_\pi(u)\,\frac{\sin 2(t-u)}{2}\,du$$

$$= \frac{\sin 2t}{2} + \frac{c}{2}\,u_\pi(t)\cdot \sin 2(t-\pi)$$

$$= \frac{\sin 2t}{2}\left(1 + c\,u_\pi(t)\right)$$

$$\Rightarrow \quad y(t) = 0 \quad \text{for} \quad t > \pi \quad \text{when} \quad c = -1.$$