Homework # 1 : Accuracy of Runge-Kutta methods.
Due on Friday February 22, 2018.

We want to explore the quality of the approximation of the solution of the following initial-value problem: $\frac{d}{dt}y(t) = f(t, y(t))$ for $t > 0$ and $y(0) = y_0$ provided by s-stage Runge-Kutta methods

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i\, k_i, \quad k_i = f(t_n + c_i\, h_h, y_n + h_n \sum_{j=1}^{s} a_{ij}\, k_j), \quad i = 1, \ldots, s.$$

1. Use the 1964 Butcher's accurarcy theorem to find out the order of accuracy of the methods explored by Runge back in 1895, namely,

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array} \qquad \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

modified Euler — improved Euler

How is the "improved" Euler method an improvement over the classic explicit Euler method?

2. Now, apply the theorem to find the order of accuracy of the following implicit Runge-Kutta methods:

$$\begin{array}{c|cc} \frac{3-\sqrt{3}}{6} & \frac{1}{4} & \frac{3-\sqrt{12}}{12} \\ \frac{3+\sqrt{3}}{6} & \frac{3+\sqrt{12}}{12} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \begin{array}{c|cc} \frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{3}{4} & \frac{1}{4} \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array}$$

Which method is more accurate?

3. (5 pts.) Suppose that $f(t, y) = \lambda\, y$, where y is a scalar-valued function and λ is a real number. Show that the Runge-Kutta associated with the generic Butcher array

$$\begin{array}{c|c} c & A \\ \hline & b \end{array}$$

is of the form $y_{n+1} = R(h\lambda)y_n$ for some scalar $R(h\lambda)$ depending on b and A, but not on c. Then argue that the method is of order p if

$$e^{h\lambda} = R(h\lambda) + \mathcal{O}(h_n^{p+1}).$$

Verify this for each of the four Runge-Kutta methods in the previous exercises. Do your results agree with the results obtained previously?

4. (5 pts.) Consider the initial-value problem

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \omega\end{bmatrix} = \begin{bmatrix}\omega\\ -\theta\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}\theta\\ \omega\end{bmatrix}(0) = \begin{bmatrix}\pi/2\\ 0\end{bmatrix}.$$

For each of the four Runge-Kutta methods fill a history of convergence table of the error in $\theta(T)$ and $\omega(T)$ for $T = \pi/3$. to compare the order of accuracy performance of the methods.

History of convergence of method RK

n	$e_{\theta,n}$	$\alpha_{\theta,n}$	$C_{\theta,n}$	$e_{\omega,n}$	$\alpha_{\omega,n}$	$C_{\omega,n}$
2		-	-		-	-
4						
8						
...						

For each value of n, take uniform time-steps $h := T/n$, compute the approximation of $(\theta(T), \omega(T))$, (θ^n, ω^n), and set $e_{\theta,n} := |\theta(T) - \theta^n|$ and $e_{\omega,n} := |\omega(T) - \omega^n|$. Let e_n be either $e_{\theta,n}$ or $e_{\omega,n}$. Then, assuming that $e_n = C\,n^{-\alpha}$, we have that $\alpha = -\ln(e_{2n}/e_n)/\ln 2$. Motivated by this, we define

$$\alpha_n := -\ln(e_{2n}/e_n)/\ln 2, \quad C_n := e_n\, n^{\alpha_n}.$$

Is our assumption about the form of the error reasonable? Does the observed approximate order of convergence matches the order of accuracy computed in the previous exercises?

What method is more efficient?