Solutions for Homework1

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Question 1. *Prove that*

$$\|fg\|_{L^1(a,b)} \le \|f\|_{L^p(a,b)}\|g\|_{L^q(a,b)}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $\|f\|_{L^p(a,b)} := (\int_a^b |f(x)|^p dx)^{\frac{1}{p}}$.

Lemma 1. *For nonnegative real number* α, β *and* $t \in [0,1]$, *we have*

$$\alpha^t\beta^{1-t} \le t\alpha + (1-t)\beta$$

Proof of the Lemma. Take $f(x) = ln(\frac{1}{x})$, then

$$f''(x) = \frac{1}{x^2} \ge 0$$

Thus f is convex, and

$$f(t\alpha + (1-t)\beta) \le tf(\alpha) + (1-t)f(\beta)$$

That is,

$$-ln(t\alpha + (1-t)\beta) \le -tln\alpha - (1-t)ln(\beta)$$

Therefore,

$$\alpha^t\beta^{1-t} \le t\alpha + (1-t)\beta$$

□

Proof of Question1. According to Lemma, we have

$$\frac{|f(x)|}{\|f\|_{L^p(a,b)}}\frac{|g(x)|}{\|g\|_{L^q(a,b)}} \le \frac{1}{p}(\frac{|f(x)|}{\|f\|_{L^p(a,b)}})^p + \frac{1}{q}(\frac{|g(x)|}{\|g\|_{L^q(a,b)}})^q$$

Hence,

$$\begin{aligned}
\frac{\|fg\|_{L^1(a,b)}}{\|f\|_{L^p(a,b)}\|g\|_{L^q(a,b)}} &= \frac{\int_a^b |f(x)g(x)|}{\|f\|_{L^p(a,b)}\|g\|_{L^q(a,b)}}dx \\
&= \int_a^b \frac{|f(x)|}{\|f\|_{L^p(a,b)}}\frac{|g(x)|}{\|g\|_{L^q(a,b)}}dx \\
&\le \int_a^b [\frac{1}{p}(\frac{|f(x)|}{\|f\|_{L^p(a,b)}})^p + \frac{1}{q}(\frac{|g(x)|}{\|g\|_{L^q(a,b)}})^q]dx (\text{Lemma } \textbf{??}) \\
&= \frac{1}{p}\frac{\int_a^b |f(x)|^p dx}{\|f\|_{L^p(a,b)}{}^p} + \frac{1}{q}\frac{\int_a^b |g(x)|^q dx}{\|g\|_{L^q(a,b)}^q} \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1
\end{aligned}$$

As a result,

$$\|fg\|_{L^1(a,b)} \le \|f\|_{L^p(a,b)}\|g\|_{L^q(a,b)}$$

□

Question 2. *Prove that $\|f\|_{L^p(a,b)}$ is a norm on $C^0(a,b)$.*

Lemma 2 (Minkowski Inequality)**.**

$$\|f\|_{L^p(a,b)} + \|g\|_{L^p(a,b)} \ge \|f+g\|_{L^p(a,b)}$$

Proof.

$$\begin{aligned}
\|f+g\|^p_{L^p(a,b)} &= \int_a^b |f(x)+g(x)|^p dx \\
&\le \int_a^b (|f(x)|+|g(x)|)|f(x)+g(x)|^{p-1}dx \\
&= \int_a^b |f(x)||f(x)+g(x)|^{p-1}dx + \int_a^b |g(x)||f(x)+g(x)|^{p-1}dx \\
&\le (\int_a^b |f(x)|^p dx)^{\frac{1}{p}} (\int_a^b |f(x)+g(x)|^{(p-1)(\frac{p}{p-1})}dx)^{1-\frac{1}{p}} \\
&\quad +(\int_a^b |g(x)|^p dx)^{\frac{1}{p}} (\int_a^b |f(x)+g(x)|^{(p-1)(\frac{p}{p-1})}dx)^{1-\frac{1}{p}} (\textit{Hölder's Inequality}) \\
&= (\|f\|_{L^p(a,b)} + \|g\|_{L^p(a,b)})\|f+g\|^{p-1}_{L^p(a,b)}
\end{aligned}$$

Hence,

$$\|f\|_{L^p(a,b)} + \|g\|_{L^p(a,b)} \ge \|f+g\|_{L^p(a,b)}$$

□

Proof of Question2. Obviously, for every $f \in C^0(a,b)$, $\|f\|_{L^p(a,b)}$ is well-defined, then we start to check that is a norm.

1. *Positive Definiteness*

   For every $f \in C^0(a,b)$, we have

   $$\|f\|_{L^p(a,b)} = (\int_a^b |f(x)|^p dx)^{\frac{1}{p}} \ge 0$$

   And for every nonzero f, exists $x_0 \in (a,b)$ such that $f(x_0) \ne 0$. For the continuity of f, exists $\delta > 0$, such that for every $x \in (x_0-\delta, x_0+\delta)$, we have $|f(x_0) - f(x)| \le \frac{1}{2}|f(x_0)|$, thus, $|f(x)| > \frac{1}{2}|f(x_0)|$.

Therefore,

$$\begin{aligned}
\|f\|_{L^p(a,b)} &= (\int_a^b |f(x)|^p dx)^{\frac{1}{p}} \\
&\geq (\int_{x_0-\delta}^{x_0+\delta} |f(x)|^p dx)^{\frac{1}{p}} \\
&\geq (\int_{x_0-\delta}^{x_0+\delta} (\frac{|f(x_0)|}{2})^p dx)^{\frac{1}{p}} \\
&= \frac{|f(x_0)|}{2}(2\delta)^{\frac{1}{p}} > 0
\end{aligned}$$

2. *Triangle Inequality* According to the Lemma,

$$\|f\|_{L^p(a,b)} + \|g\|_{L^p(a,b)} \geq \|f+g\|_{L^p(a,b)}$$

3. *Homogeneity* For $\alpha \in \mathbb{R}$,

$$\begin{aligned}
\|\alpha f\|_{L^p(a,b)} &= (\int_a^b |\alpha f(x)|^p dx)^{\frac{1}{p}} \\
&= |\alpha|(\int_a^b |f(x)|^p dx)^{\frac{1}{p}} \\
&= |\alpha|\|f\|_{L^p(a,b)}
\end{aligned}$$

□

Question 3. *Prove that*

$$\|f\|_{H^1(a,b)} := (\|f\|^2_{L^2(a,b)} + \|\frac{d}{dx}f\|^2_{L^2(a,b)})^{\frac{1}{2}}$$

is a norm on $C^1(a,b)$.

Lemma 3.

$$(\sum_{i=1}^n |a_i|^p)^{\frac{1}{p}} + (\sum_{i=1}^n |b_i|^p)^{\frac{1}{p}} \geq (\sum_{i=1}^n |a_i + b_i|^p)^{\frac{1}{p}}$$

Particularly, for $n = 1$,

$$(a^2 + b^2)^{\frac{1}{2}} \leq a + b$$

which is also a simple consequence of Cauchy-Schwartz Inequality.

Proof of the Lemma. Define

$$f(x) = \begin{cases} a_i & x \in [i-1, i), for\ i \in \{1, 2, \dots, n\}, \\ 0 & otherwise. \end{cases}$$

And

$$g(x) = \begin{cases} b_i & x \in [i-1, i), for\ i \in \{1, 2, \dots, n\}, \\ 0 & otherwise. \end{cases}$$

Apply Minkowski Inequality on f and g, we get,

$$(\sum_{i=1}^{n}|a_i|^p)^{\frac{1}{p}} + (\sum_{i=1}^{n}|b_i|^p)^{\frac{1}{p}} \geq (\sum_{i=1}^{n}|a_i + b_i|^p)^{\frac{1}{p}}$$

□

Proof of Question 3. Obviously, for every $f \in C^1(a,b)$, $\|f\|^2_{L^2(a,b)}$ and $\|\frac{d}{dx}f\|^2_{L^2(a,b)})$ is well-defined ,so $\|f\|_{H^1(a,b)}$ is well-defined, then we start to check that is a norm.

1. *Positive Definiteness*

   For every $f \in C^1(a,b)$, we have

   $$\|f\|_{H^1(a,b)} = (\|f\|^2_{L^2(a,b)} + \|\frac{d}{dx}f\|^2_{L^2(a,b)})^{\frac{1}{2}} \geq 0$$

   And for every nonzero f, $\|f\|_{L^2(a,b)} > 0$, so $\|f\|_{H^1(a,b)} > 0$.

2. *Triangle Inequality*

   $$\begin{aligned} \|f+g\|_{H^1(a,b)} &= (\|f+g\|^2_{L^2(a,b)} + \|\frac{d}{dx}(f+g)\|^2_{L^2(a,b)})^{\frac{1}{2}} \\ &\leq (\|f\|^2_{L^2(a,b)} + \|g\|^2_{L^2(a,b)} + \|\frac{d}{dx}f\|^2_{L^2(a,b)} + \|\frac{d}{dx}g\|^2_{L^2(a,b)})^{\frac{1}{2}} (Lemma\textbf{??}) \\ &\leq (\|f\|^2_{L^2(a,b)} + \|\frac{d}{dx}f\|^2_{L^2(a,b)})^{\frac{1}{2}} + (\|g\|^2_{L^2(a,b)} + \|\frac{d}{dx}g\|^2_{L^2(a,b)})^{\frac{1}{2}} (Lemma\textbf{??}) \end{aligned}$$

3. *Homogeneity* For $\alpha \in \mathbb{R}$,

   $$\begin{aligned} \|\alpha f\|_{H^1(a,b)} &= (\|\alpha f\|^2_{L^2(a,b)} + \|\frac{d}{dx}(\alpha f)\|^2_{L^2(a,b)})^{\frac{1}{2}} \\ &= (|\alpha|^2\|f\|^2_{L^2(a,b)} + |\alpha|^2\|\frac{d}{dx}f\|^2_{L^2(a,b)})^{\frac{1}{2}} \\ &= |\alpha|(\|f\|^2_{L^2(a,b)} + \|\frac{d}{dx}f\|^2_{L^2(a,b)})^{\frac{1}{2}} \\ &= |\alpha|\|f\|_{H^1(a,b)} \end{aligned}$$

□

Question 4. *Prove that*

$$|f|_{H^1(a,b)} \coloneqq \|\frac{d}{dx}f\|_{L^2(a,b)}$$

is a norm on $C^1_0(a,b)$*, the space of functions in* $C^1(a,b)$ *with compact support in* (a,b)*, but only a seminorm in* $C^1(a,b)$*.*

Proof. The proof of the triangle inequality and the homogeneity in $C^1(a,b)$ and $C^1_0(a,b)$ is similar with that in the proof of Question 2.

We need to prove $|\cdot|_{H^1(a,b)}$ is not positive definite in $C^1(a,b)$ but is positive definite in $C^1_0(a,b)$.

Consider $f(x) = 1$, $x \in (a,b)$, $f \in C^1(a,b)$, we have $|f|_{H^1(a,b)} = 0$, so $|\cdot|_{H^1(a,b)}$ is not a norm for $C^1(a,b)$.

For every $f(x) \in C_0^1(x)$, if $|f|_{H^1(a,b)} = 0$, from the positive definiteness of $\|\cdot\|_{L^2(a,b)}$, we have $\frac{d}{dx}f(x) = 0$,$x \in (a,b)$, thus, f must be a constant function on (a,b). Since f is compact supported in (a,b), we have $Suppf \subset [c,d]$ for some $[c,d] \subset (a,b)$. For every $y \in (a,c)$, $f(y) = 0$. Hence, f is the zero function. Therefore, $|\cdot|_{H^1(a,b)}$ is positive definite in $C_0^1(a,b)$. □

Question 5. *Prove that*

$$\|f\|_{H^{-1}(a,b)} \coloneqq \sup_{\phi \in C_0^\infty(a,b)} \frac{\int_a^b f(x)\phi(x)dx}{\|\frac{d}{dx}\phi\|_{L^2(a,b)}}$$

is a norm on $C^0(a,b)$.

Proof. For every $f \in C^0(a,b)$, and $\phi \in C_0^\infty(a,b)$, Let $F(x) = \int_{\frac{a+b}{2}}^x f(x)dx$, according to Hölder's Inequality we have

$$\begin{aligned} |\int_a^b f(x)\phi(x)dx| &= \left|F(x)\phi(x)|_{a+}^{b-} - \int_a^b F(x)\frac{d}{dx}\phi(x)dx\right| \\ &= |-\int_a^b F(x)\frac{d}{dx}\phi(x)| \\ &\le \int_a^b |F(x)\frac{d}{dx}\phi(x)|dx \\ &\le \|F\|_{L^2(a,b)}\|\frac{d}{dx}\phi\|_{L^2(a,b)} \end{aligned}$$

Hence, $\frac{\int_a^b f(x)\phi(x)dx}{\|\frac{d}{dx}\phi\|_{L^2(a,b)}}$ is bounded, and thus,

$$\|f\|_{H^{-1}(a,b)} \coloneqq \sup_{\phi \in C_0^\infty(a,b)} \frac{\int_a^b f(x)\phi(x)dx}{\|\frac{d}{dx}\phi\|_{L^2(a,b)}}$$

is well-defined.

1. *Positive Definiteness*

   $f \in C^0(a,b)$, if $\|f\|_{H^{-1}(a,b)} = 0$, then for every $\phi \in C_0^\infty(x)$, we have $\int_a^b f(x)\phi(x)dx = 0$. Otherwise

   $$\frac{\int_a^b f(x)\phi(x)dx}{\|\frac{d}{dx}\phi\|_{L^2(a,b)}}$$

   or

   $$\frac{\int_a^b f(x)(-\phi(x))dx}{\|\frac{d}{dx}(-\phi)\|_{L^2(a,b)}}$$

   will be greater than 0.

   Suppose f is nonzero, then exists $x_0 \in (a,b)$ such that $f(x_0) \neq 0$. For the continuity of f, exists $\delta > 0$, such that for every $x \in (x_0 - \delta, x_0 + \delta)$ and $a < x_0 - \delta < x_0 + \delta < b$, we have $|f(x_0) - f(x)| \le \frac{1}{2}|f(x_0)|$, thus, $|f(x)| > \frac{1}{2}|f(x_0)|$.

Consider $\phi(x) = \begin{cases} exp(-\frac{\delta^2}{\delta^2 - |x-x_0|^2}) & for\ |x-x_0| < \delta, \\ 0 & for\ |x-x_0| \geq \delta. \end{cases}$, we have,

$\phi(x) \in C_0^\infty(a,b)$ and $\int_a^b f(x)\phi(x)dx = \frac{|f(x_0)|}{2}\int_{x_0-\delta}^{x_0+\delta} \phi(x)dx > 0$, Contradiction!

2. *Triangle Inequality*

$$\frac{\int_a^b (f(x)+g(x))\phi(x)dx}{\|\frac{d}{dx}\phi\|_{L^2(a,b)}} = \frac{\int_a^b f(x)\phi(x)dx}{\|\frac{d}{dx}\phi\|_{L^2(a,b)}} + \frac{\int_a^b g(x)\phi(x)dx}{\|\frac{d}{dx}\phi\|_{L^2(a,b)}}$$

Taking $Sup_{\phi \in C_0^\infty(a,b)}$ on both sides, we have,

$$\|f\|_{H^{-1}(a,b)} + \|g\|_{H^{-1}(a,b)} \geq \|f+g\|_{H^{-1}(a,b)}$$

3. *Homogeneity* For $\alpha \in \mathbb{R}$, if $\alpha \geq 0$,

$$\begin{aligned} \|\alpha f\|_{H^{-1}(a,b)} &= \sup_{\phi \in C_0^\infty(a,b)} \frac{\int_a^b \alpha f(x)\phi(x)dx}{\|\frac{d}{dx}\phi\|_{L^2(a,b)}} \\ &= \alpha \sup_{\phi \in C_0^\infty(a,b)} \frac{\int_a^b f(x)\phi(x)dx}{\|\frac{d}{dx}\phi\|_{L^2(a,b)}} \end{aligned}$$

If $\alpha < 0$, then,

$$\begin{aligned} \|\alpha f\|_{H^{-1}(a,b)} &= \sup_{\phi \in C_0^\infty(a,b)} \frac{\int_a^b \alpha f(x)\phi(x)dx}{\|\frac{d}{dx}\phi\|_{L^2(a,b)}} \\ &= \sup_{\phi \in C_0^\infty(a,b)} \frac{\int_a^b (-\alpha) f(x)(-\phi(x))dx}{\|\frac{d}{dx}(-\phi)\|_{L^2(a,b)}} \\ &= (-\alpha) \sup_{\phi \in C_0^\infty(a,b)} \frac{\int_a^b f(x)\phi(x)dx}{\|\frac{d}{dx}\phi\|_{L^2(a,b)}} \end{aligned}$$

Hence,

$$\|\alpha f\|_{H^{-1}(a,b)} = |\alpha|\|f\|_{H^{-1}(a,b)}$$

□