Homework #1: Norms. Due on Friday, September 14, 2018.

1. (4 points) Prove that

$$\|f\,g\|_{L^1(a,b)} \leq \|f\|_{L^p(a,b)}\,\|g\|_{L^q(a,b)} \quad \forall\, p \in [1,\infty),$$

where $1/p+1/q=1$ and $\|f\|_{L^p(a,b)} := \big(\int_a^b |f(x)|^p\,dx\big)^{1/p}$. This is the so-called **Hölder** inequality. For $p=2$, it is called the **Cauchy-Schwarz** inequality.

Hint: Use the inequality $\alpha^t\beta^{1-t} \leq t\,\alpha+(1-t)\beta$, for any nonnegative real numbers α and β, and any $t \in [0,1]$. Prove this inequality by using the fact that the mapping $x \mapsto \ln(1/x)$ is convex on $(0,\infty)$.

2. (4 points) Prove that $\|f\|_{L^p(a,b)}$ is a norm on $C^0(a,b)$.

Hint: To prove the **Minkowski** *inequality, namely,*

$$\|f+g\|_{L^p(a,b)} \leq \|f\|_{L^p(a,b)} + \|g\|_{L^p(a,b)},$$

use the results of the previous exercise.

3. (4 points) Prove that

$$\|f\|_{H^1(a,b)} := (\|f\|^2_{L^2(a,b)} + \|\frac{d}{dx}f\|^2_{L^2(a,b)})^{1/2},$$

is a norm on $C^1(a,b)$.

4. (4 points) Prove that

$$|f|_{H^1(a,b)} := \|\frac{d}{dx}f\|_{L^2(a,b)},$$

is a norm on $C^1_0(a,b)$, the space of functions in $C^1(a,b)$ with compact support in (a,b), but only a seminorm in $C^1(a,b)$.

5. (4 points) Prove that

$$\|f\|_{H^{-1}(a,b)} := \sup_{\phi\in C_0^\infty(a,b)} \frac{\int_a^b f(x)\,\phi(x)\,dx}{\|\frac{d}{dx}\phi\|_{L^2(a,b)}},$$

is a norm on $C^0(a,b)$.

Hint: To prove that $\|f\|_{H^{-1}(a,b)} = 0$ implies $f=0$, proceed by contradiction. Assume that f is not zero and, as a consequence, that there is a point $x_0 \in (a,b)$ such that $f(x_0) \neq 0$. Obtain a contradiction by using the fact that $\|f\|_{H^{-1}(a,b)} = 0$.