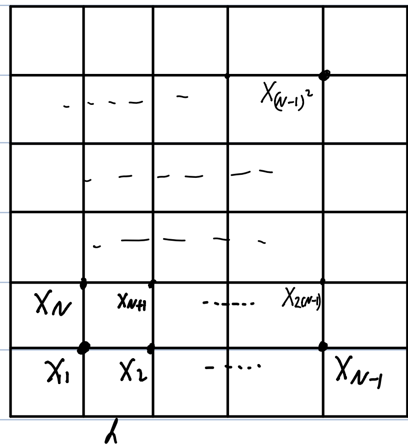

$$\begin{cases} -\Delta u = f & (0,1)^2 \\ u = 0 & \partial(0,1)^2 \end{cases}$$

$$h = \frac{1}{N}$$

1. We label the nodes row by row, namely

$$u_{ij} = (N-1)(i-1) + j \qquad 1 \le i, j \le N$$

Then the five-point finite difference method gives a matrix equation

$$Ax = b \qquad \cdots\cdots (*)$$

where

$$A = \frac{-1}{h^2} \begin{pmatrix} B & I & & \\ I & B & I & \\ & \ddots & \ddots & I \\ & & I & B \end{pmatrix}_{(N-1)^2 \times (N-1)^2}$$

and

$$B = \begin{pmatrix} -4 & 1 & & \\ 1 & -4 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -4 \end{pmatrix}_{(N-1)\times(N-1)} \qquad I = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}_{(N-1)\times(N-1)} \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{(N-1)^2 \times 1}$$

Notice that the diagonal entries of A are the same. Then $D = \frac{4}{h^2} Id$, where Id is the identity matrix of $(N-1)^2 \times (N-1)^2$.

Let $-L$ and $-U$ be the lower and upper triangular part of A.

① Jacobi method

$$x^{k+1} = (Id - D^{-1}A)x^k + D^{-1}b$$

② Gauss - Seidel method

$$x^{k+1} = (D-L)^{-1} U x^k + (D-L)^{-1} b$$

③ SOR method

$$x^{k+1} = (D-wL)^{-1}((1-w)D + wU)x^k + w(D-wL)^{-1} b$$

where $w = w_{opt} = \frac{2}{1+\sin(\pi h)}$.

For each method, we want to find the smallest k such that

$$\|x - x^k\| \leq 10^{-6} \|x - x^0\|$$

where x is the exact solution of (*), which we compute using MATLAB.

The numerical results are as follows.

N	Jacobi	Gauss-Seidel	SOR
2	1	1	1
4	40	21	11
8	175	88	22
16	712	357	45
32	2862	1432	90

2. Using the results we got in class, we know the steps we we need are

$$
\begin{aligned}
(4.34)\quad k &\approx \frac{|\ln \tau|}{\mathcal{R}(J)} \approx \frac{2|\ln \tau|}{\pi^2 h^2} \approx \frac{4|\ln h|}{\pi^2 h^2} && \text{for the Jacobi method,}\\
k &\approx \frac{|\ln \tau|}{\mathcal{R}(H_1)} \approx \frac{|\ln \tau|}{\pi^2 h^2} \approx \frac{2|\ln h|}{\pi^2 h^2} && \text{for the Gauss-Seidel method,}\\
k &\approx \frac{|\ln \tau|}{\mathcal{R}(H_{\omega_{\mathrm{opt}}})} \approx \frac{|\ln \tau|}{2\pi h} \approx \frac{|\ln h|}{\pi h} && \text{for the SOR method with optimal } \omega = \omega_{\mathrm{opt}},
\end{aligned}
$$

where τ is the tolerance.

For our numerical results, we can clearly see that the steps needed for Jacobi is twice of Gauss-Seidel. And both Jacobi and G-S method are increasing with order N^2, which is $\frac{1}{h^2}$. This is because when $N \Rightarrow 2N$, the steps are quadrupled. For the same reason, we know SOR is increasing with order N. Next let's see the theorical results:

N	Jacobi	Gauss-Seidel	SOR
2	12	6	5
4	45	23	9
8	180	90	18
16	717	359	36
32	2867	1434	71

If we compare the theorical results with our numerical results. We can see that they are very close too. So our numerical results match with the theory.

3. $x^{k+1} = (1-w)x^k + wD^{-1}(D-A)x^k + wD^{-1}b$

$= ((1-w)Id + wId - wD^{-1}A)x^k + wD^{-1}b$

$= (Id - wD^{-1}A)x^k + wD^{-1}b$

So $M_w = Id - wD^{-1}A$.

Notice that $M_w = (1-w)Id + w(Id - D^{-1}A)$ and $Id - D^{-1}A$ is the Iteration matrix of Jacobi method

Therefore, if λ is an eigenvalue of $Id - D^{-1}A$,

then $\tilde{\lambda} = (1-w) + w\lambda$ is an eigenvalue of M_w.

To make this method converges, we need

$$-1 < 1-\omega+\omega\lambda < 1$$

$$\Rightarrow \quad -2 < \omega(\lambda-1) < 0$$

$$0 < \omega(1-\lambda) < 2$$

For our model problem, the eigenvalues of $Id - D^{-1}A$ are

$$1 - \left(\sin^2\frac{m\pi h}{2} + \sin^2\frac{n\pi h}{2}\right) \qquad 1 \le m, n \le N-1$$

Hence $\quad 1-\lambda = \sin^2\frac{m\pi h}{2} + \sin^2\frac{n\pi h}{2}$

Then $\quad 0 < \omega < \dfrac{2}{\sin^2\frac{m\pi h}{2} + \sin^2\frac{n\pi h}{2}} \qquad \forall\ 1 \le m, n \le N-1$

$$\Rightarrow \quad 0 < \omega < \frac{2}{2\sin^2\frac{(N-1)\pi h}{2}}$$

Notice that

$$\frac{2}{2\sin^2\frac{(N-1)\pi h}{2}} = \frac{2}{1-\cos((N-1)\pi h)} = \frac{2}{1-\cos(\pi-\pi h)} = \frac{2}{1+\cos(\pi h)}$$

So $\quad 0 < \omega < \dfrac{2}{1+\cos(\pi h)}$

#

4. From question 3, we know

$$M = \prod_{i=1}^{l} M_{w_i}$$

$$= \prod_{i=1}^{l} \left(1 - w_i + w_i (Id - D^{-1}A)\right)$$

So if λ is an eigenvalue of $D^{-1}A$, then $\prod_{i=1}^{l}(1-w_i+w_i(1-\lambda)) = \prod_{i=1}^{l}(1-w_i\lambda)$ is an eigenvalue of M.

For $D^{-1}A$, we know the eigenvalues are

$$\lambda^{mn} = \sin^2\frac{m\pi h}{2} + \sin^2\frac{n\pi h}{2}$$

where $1 \le m, n \le N-1$.

Therefore, $\left\{\prod_{i=1}^{l}(1-w_i\lambda^{mn})\right\}_{m,n=1}^{N-1}$ are the eigenvalues of M.

Now in order to make our method converge fast than the original Jacobi method, we need to choose $\{w_i\}_{i=1}^{l}$ such that

$$\rho\left(\prod_{i=1}^{l} M_{w_i}\right) < \rho\left(\prod_{i=1}^{l} M_1\right) \quad \cdots\cdots (*)$$

To make $(*)$ hold, we can require that

$$\left|\prod_{i=1}^{l}(1-w_i\lambda^{mn})\right| < \left|\prod_{i=1}^{l}(1-\lambda^{mn})\right| \quad \cdots\cdots (*)(*)$$

for any $1 \le m, n < N-1$.

Obviously if (**) holds, (*) is true.

Notice that $1-\cos\pi h \le \lambda^{mn} \le 1+\cos\pi h$.

Let $a = 1-\cos\pi h > 0$, $b = 1+\cos\pi h$.

Now let $f(x) = \prod_{i=1}^{l}(1-w_i x)$ in (a,b)

Then $f(0)=1$. From what we saw in the error estimate of conjugate gradient method, we know Chebyshev polynomial $\dfrac{T_l\left(\frac{x-\frac{a+b}{2}}{(b-a)/2}\right)}{T_l\left(-\frac{a+b}{b-a}\right)}$ is the minimizer of the following problem

$$\min_{\substack{\deg P = l \\ P(0)=1}} \max_{x\in(a,b)} |P(x)|$$

So if we take

$$\prod_{i=1}^{l}(1-w_i x) = \frac{T_l\left(\frac{x-\frac{a+b}{2}}{(b-a)/2}\right)}{T_l\left(-\frac{a+b}{b-a}\right)}$$

Then

$$\max_{a\le\lambda\le b}\left|\prod_{i=1}^{l}(1-w_i\lambda)\right| \le \max_{a\le\lambda\le b}\left|\prod_{i=1}^{l}(1-\lambda)\right|$$

This means (**) holds.

To be more precise, if $\prod_{i=1}^{l}(1-w_i x) = \dfrac{T_l\left(\frac{x-\frac{a+b}{2}}{(b-a)/2}\right)}{T_l\left(-\frac{a+b}{b-a}\right)}$

then $\left\{\frac{1}{w_i}\right\}_{i=1}^{l}$ should be the zeros of Chebyshev polynomial $T_l\left(\frac{x-\frac{a+b}{2}}{(b-a)/2}\right)$

Hence $\frac{1}{w_i} = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(\frac{2i-1}{2l}\pi\right)$

$$\Rightarrow \quad w_i = \left(\frac{a+b}{2} + \frac{b-a}{2}\cos\left(\frac{2i-1}{2l}\pi\right)\right)^{-1} \quad (1 \le i \le l)$$

where $a = 1-\cos\pi h$ and $b = 1+\cos\pi h$.

Next let us show some numerical results to compare Jacobi method ($w_i \equiv 1$) with our weighted Jacobi method using zeros of Chebyshev polynomial.

$l=3$

Ns	Jacobi	Weighted_Jacobi
2	1	1
4	14	8
8	59	23
16	238	84
32	954	322

$l=5$

Ns	Jacobi	Weighted_Jacobi
2	1	1
4	8	4
8	35	11
16	143	33
32	573	119

$l=10$

Ns	Jacobi	Weighted_Jacobi
2	1	1
4	4	2
8	18	5
16	72	11
32	287	33