Homework #2: Classic iterative methods. Due on Friday, March 15.

In what follows, we study the performance of three classic iterative methods for numerically inverting the matrix equation

$$A\,x = b$$

associated to the five-point finite difference approximation of the model problem

$$-\Delta u = f \quad \text{in } \Omega := (0,1)^2, \qquad u = 0 \quad \text{on } \partial\Omega.$$

We use uniform Cartesian grids of squares of size $h = 1/N$.

The objective of the following two exercises is to compare the performance of the Jacobi, Gauss-Seidel and SOR (with optimal parameter ω) methods for $f = 1$ and the initial guess $x_0 = 0$.

1. (5 points) For each method and the mesh associated to $h = 1/N$, compute the number of iterations needed to reduce the Euclidean norm of the initial error 10^6 times. Do this for $N = 2, 4, 8, \ldots$.

TABLE 0.1
Number of iterations to reduce the initial error a million times

N	Jacobi	Gauss-Seidel	SOR
2			
4			
8			
...			

Hint: To be able to compute the errors, you have to compute the exact solution x first. You can use MATLAB to do that, or use your favorite iterative method to do that. Briefly state how did you choose to do it.

2. (5 points) Do your results agree with the theory? Why? Explain in detail.

3. (5 points) The weighted Jacobi method is defined as follows:

$$x^{k+1} = (1-\omega)\,x^k + \omega\,x^k_{Jacobi},$$
$$x^k_{Jacobi} = D^{-1}(-(A-D)\,x^k + b).$$

Show that the iteration matrix is $M_\omega = Id - \omega D^{-1}A$. For what values of the auxiliary parameter ω does this method converge?

4. (5 points) Now, for each set $\{\omega_i\}_{i=1}^{\ell}$, define the method whose iteration matrix is

$$M := \Pi_{i=1}^{\ell} M_{\omega_i}.$$

Find sets $\{\omega_i\}_{i=1}^{\ell}$ for which the resulting method converges faster than the original Jacobi method, that is, than the choice $\omega_i \equiv 1$, $i = 1, \ldots, \ell$. Display numerical evidence validating your findings.

Hint: In class, we obtained the eigenvalues of the matrix M_ω for $\omega = 1$. What are the eigenvalues for a general ω?