1. $f: [0,1] \longrightarrow \mathbb{R}$ be a strictly increasing continuous function. Define c_p by the minimization problem

$$\|f - c_p\|_{L^p(0,1)} = \inf_{c \in \mathbb{R}} \|f - c\|_{L^p(0,1)}$$

Argue that c_p is an optimal approximation, to f by constant functions.

Show that

$$c_p = \begin{cases} f(\frac{1}{2}) & , \text{ for } p=1, \\ \int_0^1 f(x)\,dx & , \text{ for } p=2, \\ \frac{f(0)+f(1)}{2} & , \text{ for } p=\infty. \end{cases}$$

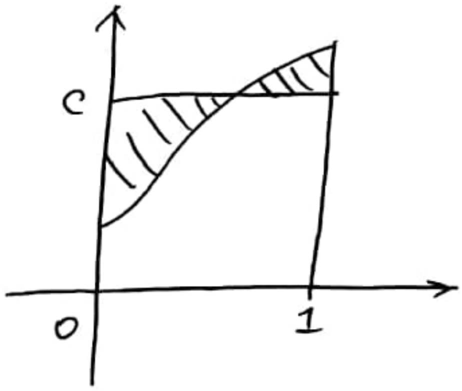
Let $g_c(x)$ be the constant function $g_c(x) = c$, and V = the space of constant functions $\subset L^p(0,1)$.

The problem is to find $\inf_{g_c \in V} \|f - g_c\|_{L^p(0,1)}$, which is an optimal approximation problem.

For $p=1$, there are several approaches to the problem,

~~Geometry~~

Geometric Approach:



$\|f-c\|_{L^1(0,1)}$

$= \int_0^1 |f(x)-c| \, dx$

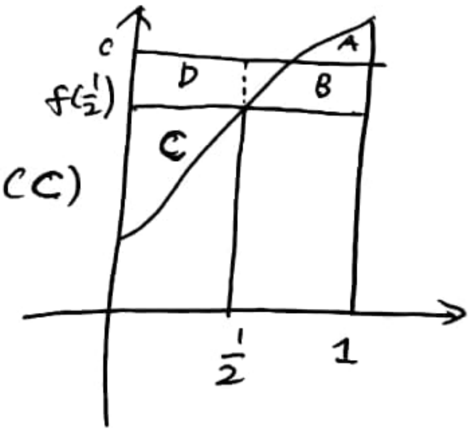
~~marked~~

is the area ~~of~~ ~~the~~ of marked part in the graph

To prove $\|f-c\|_{L^1(0,1)} > \|f-f(\frac{1}{2})\|_{L^1(0,1)}$

for $c \in (f(\frac{1}{2}), f(1))$

is to prove



$S(A) + S(C \cup D) > S(A \cup B) + S(C)$

~~that~~ is

$S(D) > S(B)$

which can be proved by $S(D) > \frac{1}{2}(c - f(\frac{1}{2})) > S(B)$.

Other cases can be proved similarly. □

There is also a metaphysical approach which has some geometric meaning

$\|f-c\|_{L^1(0,1)}$

$= \int_0^1 |f(x) - f(\frac{1}{2})|\, dx$

$= \int_{\frac{1}{2}}^1 (f(x) - f(\frac{1}{2}))\, dx + \int_0^{\frac{1}{2}} (f(\frac{1}{2}) - f(x))\, dx$ (f strictly increasing)

$= \int_{\frac{1}{2}}^1 f(x)\, dx - \int_0^{\frac{1}{2}} f(x)\, dx = \int_{\frac{1}{2}}^1 (f(x) - c)\, dx + \int_0^{\frac{1}{2}} (c - f(x))\, dx$

$\leq \int_0^1 |f(x) - c|\, dx$

The equation holds iff $f(x) \geq c$ for $x \in (\frac{1}{2}, 1)$ and $f(x) \leq c$ for $x \in (0, \frac{1}{2})$

That is $c = f(\frac{1}{2})$ for continuity of f. □

Many student want to use derivative to solve this case, but unfortunately f may not be differentiable in this problem

But we can replace $\frac{d}{dx} f(x_0)$ by

$\inf \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ in the process to obtain a Correct and Valid Solution.

For $p = 2$,

$\varphi(c) = \|f - c\|^2_{L^2(0,1)}$

$= \int_0^1 (f(x) - c)^2 \, dx$

$= c^2 - 2\int_0^1 f(x)\,dx \cdot c + \int_0^1 f^2(x)\,dx$.

$\varphi(c)$ is a polynomial of degree 2,

and it take minimal value only when

$c = \int_0^1 f(x)\,dx$.

For $p = \infty$,

for continuous function h, $\|h\|_{L^\infty(0,1)} = \sup_{x\in(0,1)} |h(x)|$

$\|f - c\|_{L^\infty(0,1)} \geq \dfrac{|f(0) - c| + |f(1) - c|}{2}$

$\geq \dfrac{|f(1) - f(0)|}{2} = \dfrac{f(1) - f(0)}{2}$

and the ~~equatil~~ equality holds

when $c = \dfrac{f(1) + f(0)}{2}$.

2. Prove that if a sequence of functions $\{\varphi_n\}_{n=1}^{\infty} \subset C^{\infty}(a,b)$ converges to some function $f: (a,b) \to \mathbb{R}$ in $L^{\infty}(a,b)$, it also converges in $L^2(a,b)$, and that if it converges in $L^2(a,b)$, it also converges in $L^1(a,b)$

Proof: (i) $\| f(x) - \varphi_n(x) \|_{L^2(a,b)}^2$

$= \int_a^b (f(x) - \varphi_n(x))^2 \, dx$

$\leq \int_a^b \left(\sup_{x \in (a,b)} |f(x) - \varphi_n(x)| \right)^2 dx = (b-a) \| f - \varphi_n \|_{L^{\infty}(a,b)}^2$

Hence,

$\| f - \varphi_n \|_{L^{\infty}(a,b)}^2 \to 0$, as $n \to \infty \Rightarrow \| f - \varphi_n \|_{L^2(a,b)}^2 \to 0$, as $n \to \infty$

(ii) $\| f(x) - \varphi_n(x) \|_{L^1(a,b)}^2$

$= \left(\int_a^b 1 \cdot | f(x) - \varphi_n(x) | \, dx \right)^2 \leq \int_a^b 1^2 \, dx \int_a^b (f(x) - \varphi_n(x))^2 \, dx$

(Hölder's Inequality)

$= (b-a) \| f - \varphi_n \|_{L^2(a,b)}^2$

Hence,

$\| f - \varphi_n \|_{L^2(a,b)} \to 0$, as $n \to \infty \Rightarrow \| f - \varphi_n \|_{L^1(a,b)} \to 0$, as $n \to \infty$.

Remark: $p < q < r$ $\| f \|_{L^q}$ can be controlled by $\| f \|_{L^p}, \| f \|_{L^r}$

$\| f \|_{L^0(a,b)} = \int_a^b 1 \, dx = b - a$

3. Assume that the sequence of functions $\{\varphi_n\}_{n=1}^{\infty} \subset C^{\infty}(a,b)$ converges to some function $f: (a,b) \to \mathbb{R}$ as follows:

$$\|f-\varphi_n\|_{L^1(a,b)} \le n^{-3}, \quad \|f-\varphi_n\|_{L^\infty(a,b)} \le n^{-2}$$

show that $\|f-\varphi_n\|_{L^2(a,b)} \le n^{-2}$.

Proof: Similar to Question 2,

$$\|f-\varphi_n\|^2_{L^2(a,b)}$$

$$= \int_a^b (f(x)-\varphi_n(x))^2\, dx = \int_a^b |f(x)-\varphi_n(x)|\,|f(x)-\varphi_n(x)|\, dx$$

$$\le \int_a^b |f(x)-\varphi_n(x)| \sup_{x\in(a,b)} |f(x)-\varphi_n(x)|\, dx$$

$$= \|f-\varphi_n\|_{L^1(a,b)} \cdot \|f-\varphi_n\|_{L^\infty(a,b)}$$

$$\le n^{-3} \cdot n^{-2} = n^{-4}.$$

Thus,

$$\|f-\varphi_n\|_{L^2(a,b)} \le n^{-2}.$$

4. Consider the functions $\varphi_n(x) := \sin(nx)$

Show that, as $n \to \infty$,

φ_n diverges to infinity in $H^1(0, 2\pi)$-norm

φ_n stay on the boundary of a ball in $L^2(0, 2\pi)$-norm

φ_n converge to 0 in $H^{-1}(0, 2\pi)$-norm

① $\|\varphi_n(x)\|_{H^1(0,2\pi)}$

$$= \left(\|\varphi_n(x)\|^2_{L^2(0,2\pi)} + \|\varphi_n'(x)\|^2_{L^2(0,2\pi)}\right)^{\frac{1}{2}}$$

$$\geq \|\varphi_n'(x)\|_{L^2(0,2\pi)} = \sqrt{\pi}\, n \to \infty, \quad \text{as } n \to \infty.$$

② $$\|\varphi_n(x)\|_{L^2(0,2\pi)} = \left(\int_0^{2\pi} \sin^2 nx \, dx\right)^{\frac{1}{2}} = \sqrt{\pi}.$$

③ $$\int_0^{2\pi} \sin nx \cdot \psi(x)\, dx = \int_0^{2\pi} \psi(x)\, d\left(-\frac{1}{n}\cos nx\right).$$

$$= \left[-\frac{1}{n}\cos nx \cdot \psi(x)\right]_0^{2\pi} + \int_0^{2\pi} \frac{1}{n} \cos nx \, d(\psi(x))$$

$$= \int_0^{2\pi} \frac{1}{n} \cos nx \, d(\psi(x)) \leq \frac{1}{n} \int_0^{2\pi} |\cos nx| |\psi'(x)| \, dx$$

$$\leq \frac{1}{n} \|\psi'\|_{L^2(0,2\pi)} \|\cos nx\|_{L^2(0,\pi)}$$

Hence

$$\| \varphi_n(x) \|_{H^{-1}(0,2\pi)} = \sup_{\psi \in C_0^\infty(0,2\pi)} \frac{\int_0^{2\pi} \sin nx \cdot \psi(x)\,dx}{\| \psi'(x) \|_{L^2(0,2\pi)}}$$

$$\leq \frac{1}{n} \sup_{\psi \in C_0^\infty} \| \cos nx \|_{L^2} \to 0 \quad \text{as } n \to \infty. \quad \square$$