**Homework #2: Convergence.** Due on Friday, September 21, 2018.

1. (5 points) Let $f : (0,1) \mapsto \mathbb{R}$ be a strictly increasing, continuous function. Define $c_p$ by the minimization problem

$$\|f - c_p\|_{L^p(0,1)} = \inf_{c\in\mathbb{R}} \|f - c\|_{L^p(0,1)}.$$

Argue that $c_p$ is an optimal approximation to $f$ by constant functions. Show that

$$c_p = \begin{cases} f(1/2) & \text{if } p = 1, \\ \int_0^1 f(x)\,dx & \text{if } p = 2, \\ (f(0) + f(1))/2 & \text{if } p = \infty. \end{cases}$$

2. (5 points) Prove that if a sequence of functions $\{\varphi_n\}_{n=1}^{\infty} \subset \mathcal{C}^{\infty}(a,b)$ converges to some function $f : (a,b) \mapsto \mathbb{R}$ in $L^{\infty}(a,b)$, it also converges in $L^2(a,b)$, and that if it converges in $L^2(a,b)$, it also converges in $L^1(a,b)$.

3. (5 points) Assume that the sequence of functions $\{\varphi_n\}_{n=1}^{\infty} \subset \mathcal{C}^{\infty}(a,b)$ converges to some function $f : (a,b) \mapsto \mathbb{R}$ as follows:

$$\begin{aligned} \|f - \varphi_n\|_{L^1(a,b)} &\leq n^{-3}, \\ \|f - \varphi_n\|_{L^{\infty}(a,b)} &\leq n^{-1}. \end{aligned}$$

Show that

$$\|f - \varphi_n\|_{L^2(a,b)} \leq n^{-2}.$$

4. (5 points) Consider the functions $\varphi_n(x) := \sin(n\,x)$. Show that, as $n$ goes to infinity, the functions $\varphi_n$ diverge to infinity in the $H^1(0,2\pi)$-norm, stay on the boundary of a ball of $L^2(0,2\pi)$, and converge to zero in $H^{-1}(0,2\pi)$.