Homework #3: Finite difference methods for the transport equation.
Due on Friday, April 5, 2019.

In what follows, u is the solution of the initial-value problem:

$$u_t + a\, u_x = 0 \text{ in } (0, 2\pi) \times (0, T), \qquad u(x, 0) = u_0(x) \text{ for all } x \in (0, 2\pi),$$

with periodic boundary conditions. We assume that $a > 0$.

We consider the following finite difference method:

$$\frac{1}{\Delta t}(u_j^{n+1} - u_j^n) + \frac{a}{\Delta x}(\widehat{u}_{j+1/2}^n - \widehat{u}_{j-1/2}^n) = 0 \qquad j = 0, ..., M-1, n = 0, ..., N-1,$$

$$u_j^0 = \frac{1}{2}(u_0(x_j^-) + u_0(x_j^+)) \qquad j = 0, ..., M-1,$$

where $x_j := j\Delta x$, with periodic boundary conditions, where $\Delta x = 2\pi/M, N := T/\Delta t$ and

$$\widehat{u}_{j+1/2}^n := \frac{1}{2}(u_{j+1}^n + u_j^n) - \frac{\alpha}{2}(u_{j+1}^n - u_j^n).$$

Here α is a real number.

1. (5 pts.) For what values of the parameter α this scheme is first-order accurate, second-order accurate and third-order accurate. Find the order of accuracy of the upwinding, downwinding, centered, Lax-Friedrichs and Lax-Wendroff schemes.

Recall that the order of accuracy of the scheme is p if the local truncation error LTE_i^n is of order $(\Delta x)^p$ whenever ν is kept constant (so that $\Delta t = \frac{\nu}{a}\Delta x$). Here

$$LTE_j^n := \frac{1}{\Delta t}(u(t^{n+1}, x_j) - u(t^n, x_j)) + \frac{a}{\Delta x}(\widehat{u}_{j+1/2}^n - \widehat{u}_{j-1/2}^n),$$

where

$$\widehat{u}_{j+1/2}^n := \frac{1}{2}(u(t^n, x_{j+1}) + u(t^n, x_j)) - \frac{\alpha}{2}(u(t^n, x_{j+1}) - u(t^n, x_j)),$$

and u is the exact solution.

2. (5 pts.) Use Fourier techniques to find all the values of α for which the method is L^2-stable and display the condition on $\nu := a\,\Delta t/\Delta x$ under which the method is L^2-stable. Are the upwinding, downwinding, centered, Lax-Friedrichs and Lax-Wendroff schemes L^2-stable?

Hint: Use the fact that the scheme is L^2-stable if the modulus of its amplification factor $g(\theta, \nu)$ is less or equal to one for $\theta \in [-\pi, \pi]$.

3. (5 pts.) Next, we explore the behavior of the centered method $\alpha = 0$. Fill the history of convergence table below for $T = 2\pi$ and $u_0(x) := \sin x$ and then for $T = 2\pi$ and $u_0(x) := 1$ on $(.8\pi, 1.2\pi)$ and $u_0(x) := 0$ elsewhere. Take $a = 1$. Make sure to maintain ν constant as you refine the space and time grids.

Table 0.2
History of convergence of the approximation given by the centered method

M	$\|u(T) - u_h(T)\|_h$	α_h	C_h
2		-	
4			
8			
...			

This means that, given an integer M, we must set $\Delta x = 2\pi/M$ and then $\Delta t = \nu\Delta x$. Since $T/\Delta t$ might not be an integer, the very last time step used in the scheme has to be modified as follows. If $N-1$ is the biggest integer less than $T/\Delta t$, the size of the last time step has to be $T - (N-1)\Delta t$.

We use the following definition:

$$\|u(T) - u_h(T)\|_h^2 := \sum_{j=0}^{M-1} (\frac{1}{2}(u(x_j^-, T) + u(x_j^+, T)) - u_j^N)^2 \,\Delta x.$$

Do your results agree with your results in the previous exercise?

4. (5 pts.) Next, we explore the behavior of the upwinding ($\alpha = 1$) and Lax-Wendroff ($\alpha = \nu$). For each of these methods, fill a history of convergence table for $T = 2\pi$ and $u_0(x) := \sin x$ and then for $T = 1$ and $u_0(x) := 1$ on $(.8\pi, 1.2\pi)$ and $u_0(x) := 0$ elsewhere. Which of these two methods is 'better'?

Take $a = 1$. Make sure to maintain ν constant as you refine the space and time grids.

This means that, given an integer M, we must set $\Delta x = 2\pi/M$ and then $\Delta t = \nu\Delta x$. Since $T/\Delta t$ might not be an integer, the very last time step used in the scheme has to be modified as follows. If $N-1$ is the biggest integer less than $T/\Delta t$, the size of the last time step has to be $T - (N-1)\Delta t$.

For the case of a smooth solution, do you obtain an order of convergence which is the same as the order of accuracy obtained in the first exercise? **Optional**: Can you explain the order of convergence obtained for the discontinuous exact solution?