1 Prove Property (iv) holds with $M = 1$.

We need to prove $\|(Id - \Pi_n)v\|_{L^2} \le \|v\|_{L^2}$.

Equivalently, $\|(Id - \Pi_n)v\|_{L^2}^2 \le \|v\|_{L^2}^2$ .... (1)

LHS of (1) − RHS of (1)

$$= \int_0^1 (v(x) - \Pi_n v(x))^2 dx - \int_0^1 v^2(x)\, dx$$

$$= \int_0^1 \Pi_n v^2(x)\, dx - 2\int_0^1 \Pi_n v(x) \cdot v(x)\, dx$$

$$= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (\Pi_n v^2(x) - 2\Pi_n v(x) \cdot v(x))\, dx$$

(by def)

$$= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} - \Pi_n v^2(x)\, dx \le 0.$$

Therefore, $\|(Id - \Pi_n)v\|_{L^2} \le \|v\|_{L^2}$. □

2. Complete the history of convergence for $v(x) = \sin \pi x$. Was the assumption $e_n = Cn^{-\alpha}$ reasonable? Do your result match the theory?

Clue for programing: 2.3.4.

$k = 1, 2, 3 \dots$

$n = 2^k$

$S = 0$

$$\left[\begin{array}{l} m = 0, 1, \dots, 2^k - 1 \\ S = S + \underbrace{\int_{\frac{m}{n}}^{\frac{m+1}{n}} (v(x) - \Pi_n v(x))^2 dx}_{} \end{array}\right.$$

calculate the explicit formula to get more accurate result

$e_n = \text{sqrt}(S)$

$$\alpha_n = -\frac{\ln(e_n / e_{n/2})}{\ln 2}$$

$C_n = e_n n^{\alpha_n}$

Table

N	e_n	α_n	C_n
2	0.3078	—	—
4	0.1587	0.9557	0.5969
8	0.0800	0.9889	0.6250
16	0.0401	0.9972	0.6359
32	0.0200	0.9993	0.6396

The Assumption is Reasonable and match the theory on the note

$\theta = 1$

(n^{-1} - convergence)

3. Complete the history of Convergence Table for $v(x) = x^{\frac{1}{5}}$

Was the assumption reasonable?

Compute the value of θ we can take in Rate-Convergence Theorem

Contrast the numerical and theoretical results.

Table

N	e_n	α_n	C_n
2	0.0906	—	—
4	0.0573	0.6601	0.1431
8	0.0359	0.6746	0.1460
16	0.0224	0.6837	0.1488
32	0.0139	0.6894	0.1512

as N get greater

$\alpha_n \to 0.7$

The Assumption is Reasonable

$\alpha \sim 0.7$

Define $\varphi_\epsilon(x) = \begin{cases} \epsilon^{\frac{1}{5}}(1+\frac{1}{5}(\frac{x}{\epsilon}-1)) & \text{for } x \in [0,\epsilon] \\ x^{\frac{1}{5}} & \text{for } x \in (\epsilon, 1] \end{cases}$

$$\|v-\varphi_\epsilon\|_{L^2} \le \left(\int_0^\epsilon (\epsilon^{\frac{1}{5}})^2 dx\right)^{\frac{1}{2}} = \epsilon^{\frac{7}{10}},$$

$$\|\tfrac{d}{dx}\varphi_\epsilon\|_{L^2} = \left(\int_0^\epsilon (\tfrac{1}{5}\epsilon^{-\frac{4}{5}})^2 dx + \int_\epsilon^1 (\tfrac{1}{5}x^{-\frac{4}{5}})^2 dx\right)^{\frac{1}{2}} \le \epsilon^{-\frac{3}{10}}$$

$$K(v,t) \le \inf_{\epsilon\in(0,1)} (\|v-\varphi_\epsilon\|_{L^2} + t\|\tfrac{d}{dx}\varphi_\epsilon\|_{L^2}) \le \inf_{\epsilon\in(0,1)} (\epsilon^{\frac{7}{10}} + t\epsilon^{-\frac{3}{10}})$$

For $t \le 1$ take $t=\epsilon$, upper bound $= 2t^{\frac{7}{10}}$

For $t > 1$ set g_c be be constant function $g_c(x) = c$,

$$K(v,t) \le \inf_{c\in\mathbb{R}} \|v-g_c\|_{L^2} = \sqrt{\tfrac{5}{2\sqrt{2}}}.$$

$$|v|_\theta \le \max\{\sup_{t\le 1} 2t^{\frac{7}{10}-\theta}, \sup_{t>1} t^{-\theta}\sqrt{\tfrac{5}{2\sqrt{2}}}) = 2$$

Choose $\theta = \frac{7}{10}$, we can get a $n^{-\frac{7}{10}}$ - convergence From Rate Convergence Theorem

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4. Complete the history of convergence table above for $u(x) = x^{\frac{1}{2}}$. Was the assumption $e_n = Cn^{-\alpha}$ reasonable? What rate of convergence is suggested by your numerical experiment?

Table

n	e_n	α_n	C_n
2	0.1320	—	—
4	0.725	0.8649	0.2404
8	0.0392	0.8858	0.2475
16	0.0210	0.9013	0.2556
32	0.0112	0.9132	0.2642

Reasonable

At Rate of

$n^{-(1-\epsilon)}$ ~~convergence~~

($1-\epsilon$ convergence).

You can make any guess of the rate of convergence from the experiment without losing points.

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