Homework #3: Rates of convergence. Due on Friday, September 28, 2018.

Let us take $(V, \| \|) := (L^2(0,1), \| \cdot \|_{L^2(0,1)})$. Then, let us partition $(0,1)$ into the intervals $I_i := (x_{i-1}, x_i)$, $i = 1, \ldots, n$, where $x_i := i/n$,and set $W_n := \{w \in V : w|_{I_i}$ is a constant, $i = 1, \cdots, n\}$. Finally, define the operator $\pi_n : V \to W_n$ by

$$\pi_n v(x) = n \int_{I_i} v(s)\, ds \quad \forall\, x \in I_i, \quad i = 1, \cdots, n.$$

We are going to explore numerically the convergence properties of the optimal approximation $\pi_n v$. Write a code to compute the approximation error $e_n := \|v - \pi_n v\|_{L^2(0,1)}$ for the three specific functions in the exercises below. Carry out the integrals on each of the elements by hand and then write the corresponding code. We do not want to use quadrature rules to approximate the integrals.

Assuming that $e_n = C\, n^{-\alpha}$, we have that $\alpha = -\ln(e_n/e_{n/2})/\ln 2$. Motivated by this, we set

$$\alpha_n := -\ln(e_n/e_{n/2})/\ln 2, \quad C_n := e_n\, n^{\alpha_n},$$

and consider the history of convergence table defined below.

Table 0.1
History of convergence of the approximation error $e_n := \|v - \pi_n v\|_{L^2(0,1)}$

n	e_n	α_n	C_n
2		-	-
4			
8			
...			

1. (5 points) We know that Property (ii) of the rates-of-convergence Theorem holds with $M_n = n^{-1}$, $|\cdot| = \|\frac{d}{dx} \cdot\|_{L^2(0,1)}$ and $D := \mathcal{C}^1[0,1]$. Prove that Property (iii) holds with $M = 1$.

2. (5 points) Complete the history of convergence table above for $v(x) = \sin(\pi\, x)$. Was the assumption $e_n = C\, n^{-\alpha}$ reasonable? Do your results match the theory?

3. (5 points) Complete the history of convergence table above for $v(x) = x^{1/5}$. Was the assumption $e_n = C\, n^{-\alpha}$ reasonable? Compute the value of θ we can take in the rates-of-convergence Theorem. Contrast your numerical and theoretical results.

4. (5 points) Complete the history of convergence table above for $v(x) = x^{1/2}$. Was the assumption $e_n = C\, n^{-\alpha}$ reasonable? What rate of convergence is suggested by your numerical experiments?