Homework #4: DG methods for the transport equation. Due on Friday, April 26, 2019.

In this exercise, we are going to get acquainted with the space discretization provided by the discontinuous Galerkin (DG) method. We do that on the following simple model problem:

$$\begin{aligned} u_t + c\, u_x &= 0 \qquad && \text{in } (0,1)\times(0,T), \\ u(t=0) &= u_0 && \text{in } (0,1), \end{aligned}$$

whrere $c>0$ and with *periodic* boundary conditions for the first three problems. For the last problem, we take $u(x=0,t)=\sin^2(t)$ for $t\in(0,T)$.

1. (5 points) Assume that $\{x_j\}_{j=0}^N$ is a partition of $[0,1]$. Set $I_j=(x_{j-1},x_j)$, $\Delta_j = x_j - x_{j-1}$, and $x_{j-1/2}=(x_j+x_{j-1})/2$, for $j=1,\dots,N$.

For each $t\ge 0$ and $x\in I_j$, take the approximate solution u_h as

$$u_h(x,t) = \sum_{\ell=0}^{k} u_j^\ell(t)\, P^\ell((x-x_{j-1/2})/(\Delta_j/2)),$$

where P^ℓ is the Legendre polynomial of degree ℓ.

Show that the equations for the degrees of freedom given by the DG-space discretization (using the upwinding flux) are

$$\frac{d}{dt}u_j^\ell + \frac{(2\ell+1)\,c}{\Delta_j}\left(\sum_{m=0}^{\ell-1}(-1)^{\ell+m}u_j^m + \sum_{m=\ell}^{k} u_j^m - \sum_{m=0}^{k}(-1)^\ell u_{j-1}^m\right) = 0.$$

Hint: Use the weak formulation and the properties

$$\int_{-1}^{1} P^\ell\, P^m = \frac{2}{2\ell+1}\,\delta_{\ell m} \qquad \text{and} \qquad P^\ell(\pm 1) = (\pm 1)^\ell.$$

2. (5 points) Show that we have

$$\frac{1}{2}\frac{d}{dt}\|\, u_h(t)\,\|^2_{L^2(\Omega)} + \Theta_h(t) = 0,$$

where

$$\Theta_h(t) = \frac{c}{2}\sum_{j=0}^{N-1}(u_h(x_j^-,t) - u_h(x_j^+,t))^2.$$

Conclude that the method is stable in $L^\infty(L^2)$, *provided* that the initial data is the L^2-projection of u_0 into the space V_h of functions that are polynomials of degree k on each interval I_j, $j=1,\dots,N$.

Hint: Take the test function equal to u_h in the weak formulation.

3. (5 points) Run the RKDG method with a polynomial approximations of degree k and an (SSP) RK method of order $k+1$ for the initial data

$$u_0(x) = \sin(2\pi\, x),$$

until $T = 0.5$. Take uniform meshes of $20, 40, 80, \dots$ elements to study the actual history of convergence of the approximation in the L^2 and in the L^∞ norms. Verify that the order of convergence is always $k+1$ for $k = 0, 1, 2, 3, 4, 5$.

Note: Use the code in the tar-file dg_transport.tar. After extracting all the files therein, type source work. Then run the code by typing run. The results are in the files res (history of convergence), apprx (approximate solutions), and exact (exact solution). The files apprx and exact are for plotting purposes only.

4. (5 points) Modify the code for the DG method to incorporate the boundary condition at $x = 0$. Display convincing numerical evidence that your code is running well for $k = 0, 1, 2, 3, 4, 5$.

Note: The code is written in FORTRAN 77. If you do not know this language, you can rewrite the code I sent you into MATLAB. Otherwise, you only have to modify the subroutine " bdryflux.f ".