Homework 4

1. Find Conditions on $\varphi \in L^2(0,\pi)$ for which we can write $\|T^{-\frac{m}{2}}\varphi\|_{L^2(0,2\pi)} = \|\frac{d^m}{dx^m}\varphi\|$

Solution: Recall the definition of T, for $f \in L^2(0,\pi)$,

Tf is defined as the solution of

$$-\frac{d^2}{dx^2}\phi = f \text{ in } (0,\pi) \text{ and } \phi = 0 \text{ on } \{0,\pi\}$$
$$\phi \in C^2[0,\pi],$$

Conversely, for $\phi = 0$ on $\{0,\pi\}$, $T^{-1}\phi = f = -\frac{d^2}{(dx)^2}\phi$

② To define $T^{-k-1}\phi$, we need $T^{-k}\phi = 0$ on $\{0,\pi\}$,

$$T^{-k-1}\phi = T^{-1}(T^{-k}\phi)$$

that is, $(-1)^k \frac{d^{2k}}{(dx)^{2k}}\phi = 0$ on $\{0,\pi\}$.

So we need to assume $\frac{d^{2k}\varphi}{(dx)^{2k}} = 0$ on $\{0,\pi\}$ $\wedge$ $\varphi \in C^{2k+2}(0,\pi)$

③ Recall T is a Hilbert-Schmidt Operator with symmetric, L^2-finite, continuous kernel,

we have T is a compact, self-adjoint operator

$$\langle T\varphi, \psi\rangle = \langle \varphi, T\psi\rangle$$

Also, $T^{\frac{1}{2}}$ is well-defined by the spectral Theorem

By verification of eigenfunctions, we have

$$\langle T^{\frac{1}{2}}\varphi, \psi\rangle = \langle \varphi, T^{\frac{1}{2}}\psi\rangle$$

(4) For given m, consider the condition

$$P(m): \frac{d^{2k}}{dx^{2k}}\varphi = 0 \text{ at } \{0,\pi\}, \text{ for } 2k < m$$

$\wedge$

$\varphi \in C^{m+1}(0,\pi)$, ~~for 2k < m~~.

Claim: $P(m) \Rightarrow \|T^{-\frac{m}{2}}\varphi\|_{L^2(0,\pi)} = \|\frac{d^m}{dx^m}\varphi\|_{L^2(0,\pi)}$

Pf: (1) For even m,

$$T^{-\frac{m}{2}}\varphi = (-1)^{\frac{m}{2}}\frac{d^m}{(dx)^m}\varphi,$$

our claim is obvious.

(2) For odd m,

$$(T^{-\frac{m}{2}}\varphi, T^{-\frac{m}{2}}\varphi) = (T^{-\frac{m-1}{2}}\varphi, T^{-\frac{m+1}{2}}\varphi)$$

well defined since $\frac{d^{m-1}}{dx^{m-1}}\varphi = 0$ at $\{0,\pi\}$

$$= -\int_0^\pi \varphi^{(m-1)}\cdot\varphi^{(m+1)}dx = -\varphi^{(m-1)}\varphi^{(m)}\Big|_0^\pi + \int_0^\pi \varphi^{(m)}\cdot\varphi^{(m)}dx$$

$$= \langle \varphi^{(m)}, \varphi^{(m)}\rangle = \langle \frac{d^m}{dx^m}\varphi, \frac{d^m}{dx^m}\varphi\rangle$$

$\square$

Rmk: $P(m) \Leftrightarrow$ Q1, that is exactly what we need.

2 & 3. $v(x) := 1 - \cos(2x)$. Apply our estimates with $s = \frac{3}{2}$ $(s = 3)$ to obtain a rate of convergence of $\|v - \Pi_n v\|_{L^2(0,\pi)}$ where Π_n is the L^2-orthogonal projection into the space $W_n := \operatorname{span}\{\sin(ix),\ i = 1, \cdots, n\}$.

Solution. We will do the estimate for integer $m \geq 3$ and $s = \frac{m}{2}$.

Define φ_ϵ with an indetermined polynomial $p(x)$,

$$\varphi_\epsilon(x) = \begin{cases} v(x) + \epsilon^2 p\left(\frac{x}{\epsilon}\right), & \text{for } 0 \leq x \leq \epsilon, \\ v(x), & \text{for } \epsilon < x \leq \frac{\pi}{2}, \\ \varphi_\epsilon(\pi - x), & \text{for } \frac{\pi}{2} < x \leq \pi. \end{cases}$$

We want ① $\varphi_\epsilon \in C^{m+1}[0,\pi]$, that is,

$\chi_{[0,\epsilon]}(x) p\left(\frac{x}{\epsilon}\right) \in C^{m+1}[0,\pi]$,

② For $2k \leq m$, $\varphi_\epsilon^{(2k)}(x) = 0$ at $x = 0, \pi$,

that is $p^{(2k)}(0) = 0$.

Take $p(x) = x^{m+1}(1-x)^{m+1} p_0(x)$, these conditions can be satisfied.

$$\|\varphi_\epsilon - v\|_{L^2(0,1)} = 2\,\|\epsilon^2 p(\tfrac{x}{\epsilon})\|_{L^2[0,\epsilon]}$$

$$= 2\epsilon^2\left(\int_0^\epsilon (p(\tfrac{x}{\epsilon}))^2 dx\right)^{\frac{1}{2}}$$

$$= 2\epsilon^{\frac{5}{2}}\left(\int_0^1 (p(\tfrac{x}{\epsilon}))^2 d\tfrac{x}{\epsilon}\right)^{\frac{1}{2}}$$

$$= C_0\,\epsilon^{\frac{5}{2}}$$

From Question 1 and condition ②,

$$\|T^{-s}\varphi_\epsilon\| = \left\|\frac{d^m\varphi_\epsilon}{dx^m}\right\| \le \left\|\frac{d^m v}{dx^m}\right\| + 2\left\|\epsilon^2\frac{d^m p(\frac{x}{\epsilon})}{dx^m}\right\|_{L^2}$$

$$\cancel{\left\|\frac{d^m\varphi_\epsilon}{dx^m}\right\|} - \quad = C_1 + C_2\cdot\epsilon^{\frac{5}{2}-m}$$

Since $m \ge 3$, $\epsilon^{\frac{5}{2}-m} \to +\infty$ as $\epsilon \to 0^+$.

$\exists\ C_3$ s.t. $C_1 + C_2\cdot\epsilon^{\frac{5}{2}-m}$, for $0<\epsilon\le\frac{\pi}{2}$.

~~Similarly, $\|T^{-s}\varphi_\epsilon\| = \|\frac{d^m\varphi_\epsilon}{dx^m}\| \ge 2\|\epsilon^2\frac{d^m p(\frac{x}{\epsilon})}{dx^m}\| - \|\frac{d^m v}{dx^m}\|$~~

$$K(v,t) = \inf\ C_0\epsilon^{\frac{5}{2}} + tC_3\cdot\epsilon^{\frac{5}{2}-m}.$$

Take $\epsilon = t^{\frac{1}{m}}$, $K(v,t) \le C\,t^{\frac{5}{2m}}$.

Take $\theta = \frac{5}{2m}$, $|v|_{\theta,s} \le C$

From Theorem 3.5,

$$\|v - \pi_n v\| \le k_{n+1}^{s\theta}\,|v|_{\theta,s} = C\cdot(n+1)^{-\frac{5}{2}}$$

□

4. Fill this History ~~of~~ Convergence table below. Do~~es~~ the results match the theoretical prediction?

n	e_n	α_n	C_n
2	0.431	—	—
4	0.065	2.7~~3~~26	2.849
8	0.011	2.5~~4~~36	2.187
16	0.002	2.547	2.061

Apply the program (PLZ ~~don't took~~ ignore the sample) 

for the last homework, and get the table above.

It match the Convergence Rate is around $Cn^{-\frac{5}{2}}$