Homework #6. Complete sets. Due on Friday, October 26, 2018.

In this homework, we are going to use the following result.

THEOREM 0.1. *Let V be a separable Hilbert space, and let $T : V \to V$ be an operator such that*

(i) *T is linear, bounded, self-adjoint, and positive-definite,*

(ii) *$\{\phi_j\}_{j=1}^{\infty} \subset V$ is the set of all eigenvectors of T: $T\phi_j = \kappa_j\,\phi_j$,*

(iii) *$\kappa_j \to 0$ as j goes to infinity.*

Then, $\{\phi_j\}_{j=1}^{\infty}$ is an orthogonal basis of V.

1. (5 points) Use the above theorem to show that the set $\{\sin jx\}_{j=1}^{\infty}$ is complete in $L^2(0,\pi)$. The inner product is $\langle f_1, f_2\rangle := \int_0^{\pi} f_1(x)\, f_2(x)\, dx$.

Hint: Consider the mapping $f \mapsto Tf := \varphi$, where φ is the solution of the boundary value problem

$$-\frac{d^2}{dx^2}\varphi = f \quad \textit{in } (0,\pi), \quad \varphi(x) = 0 \quad \textit{for } x \in \{0,\pi\}.$$

Then, verify the three assumptions of the theorem. Recall that the linear operator $L : V \to V$ is bounded if we can find a constant M such that $\|Lv\| \le M\|v\|$ for all $v \in V$.

2. (5 points) Use the above theorem to show that the set $\{\sin jx\}_{j=1}^{\infty}$ is complete in $H_0^1(0,\pi) := \{v \in L^2(0,\pi) : \frac{d}{dx}v \in L^2(0,\pi),\ v = 0 \text{ on } \{0,\pi\}\}$. The inner product is $\langle f_1, f_2\rangle := \int_0^{\pi} \frac{d}{dx}f_1(x)\ \frac{d}{dx}f_2(x)\, dx$.

Hint: Consider the same operator T considered in the first problem. Note that now $V := H_0^1(0,\pi)$ and that the norm of f, $\|f\|$, is $\|\frac{d}{dx}f\|_{L^2(0,\pi)}$.

3. (5 points) Let $T : V \to V$ be any operator satisfying the assumptions (i) and (ii) of the above Theorem where $V = L^2(a,b)$. Assume that we have that

$$Tf(x) = \int_a^b \mathsf{G}(x,y)\, f(y)\, dy.$$

Prove that assumption (iii) is true if $\int_a^b \int_a^b \mathsf{G}^2(x,y)\, dx\, dy < \infty$.

Hint: For any fixed value of x, consider the function $y \mapsto \mathsf{G}(x,y)$. Let $\pi_n\mathsf{G}(x,\cdot)$ be its orthogonal projection into $W_n := \mathsf{span}\{\phi_j\}_{j=1}^{n}$. Use Bessel's inequality, argue that $\langle \mathsf{G}(x,\cdot), \phi_j\rangle = T\phi_j(x)$, and integrate over x to obtain that $\sum_{j=1}^{n} \kappa_j^2 \le \int_a^b \int_a^b \mathsf{G}^2(x,y)\, dx\, dy$.

4. (5 points) Let u be the solution of

$$-\frac{d^2}{dx^2}u = f \text{ in } (0,\pi), \qquad u(x) = 0 \text{ for } x \in \{0,\pi\}.$$

Let $P_n u$ be the element of the space $W_n := \text{span}\{\sin jx : j = 1, \ldots, n\}$, which satisfies the equation

$$\int_0^\pi \frac{d}{dx} P_n u(x) \, \frac{d}{dx} \sin jx \, dx = \int_0^\pi f(x) \, \sin jx \, dx \quad j = 1, \ldots, n.$$

Argue that $P_n u$ is the orthogonal projection of u into the space $V := H_0^1(0, \pi)$ with inner product $\langle f_1, f_2 \rangle := \int_0^\pi \frac{d}{dx} f_1(x) \, \frac{d}{dx} f_2(x) \, dx$. Show that $P_n u$ is **coincides** with the orthogonal $\pi_n u$ projection of u into W_n with $V := L^2(0, \pi)$ and inner product $\langle f_1, f_2 \rangle := \int_0^\pi f_1(x) \, f_2(x) \, dx$.