Homework #7

(1) The error of the quadrature rule $Q(u) = \sum_{l=1}^{m} w_l u(x_l)$ is

$$e(u) := \int_a^b u(x)\,dx - \sum_{l=1}^{m} w_l u(x_l)$$

If the quadrature rule is exact for polynomials of degree i, we can write

$$e(u) = \int_a^b K(s) \frac{d^{(i+1)}}{ds^{i+1}} u(s)\,ds$$

where the Peano kernel is

$$K(s) = \int_a^b \frac{(x-s)_+^i}{i!}\,dx - \sum_{l=1}^{m} w_l \frac{(x_l - s)_+^i}{i!}$$

$$= e\left(\frac{(\cdot - s)_+^i}{i!} \right)$$

Both the mid-point rule as well as the trapezoidal rule have degree of exactness equal to one. So, we can apply the above formula with $i=1$. We get

$$K_M(s) = \int_a^b (x-s)_+\,ds - (b-a)\left(\tfrac{a+b}{2} - s\right)_+$$

$$K_T(s) = \int_a^b (x-s)_+ - \frac{b-a}{2}\left((a-s)_+ + (b-s)_+\right)$$

then

$$K_M(s) = \frac{(b-s)^2}{2} - \begin{cases} (b-a)\left(\frac{a+b}{2} - s\right) & s \in \left(a, \frac{a+b}{2}\right), \\ 0 & s \in \left(\frac{a+b}{2}, b\right), \end{cases}$$

$$= \begin{cases} \frac{1}{2}(b-s)^2 - ((b-s)+(s-a))\left(\frac{a-s}{2} + \frac{b-s}{2}\right) & s \in \left(a, \frac{a+b}{2}\right), \\ \frac{1}{2}(b-s)^2 & s \in \left(\frac{a+b}{2}, b\right), \end{cases}$$

$$= \begin{cases} \frac{1}{2}(s-a)^2 & s \in \left(a, \frac{a+b}{2}\right), \\ \frac{1}{2}(b-s)^2 & s \in \left(\frac{a+b}{2}, b\right), \end{cases}$$

and

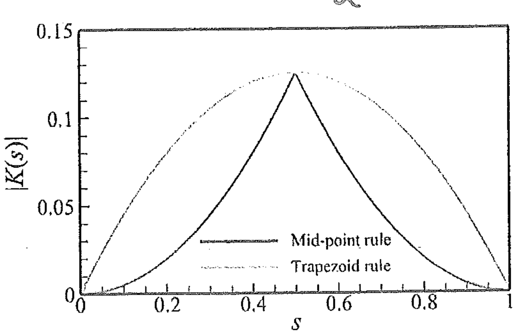
$$K_T(s) = \frac{(b-s)^2}{2} - \frac{b-a}{2}(b-s)$$

$$= \frac{1}{2}(b-s)(b-s-b+a)$$

$$= \frac{1}{2}(b-s)(a-s)$$

Since $|e(u)| \le \int_a^b |K(s)|\, ds \cdot \left\| \frac{d^2}{dx^2} u \right\|_{L^\infty(a,b)}$, we expect the midpoint rule to be, in general, more accurate than the trapezoidal rule since

$$\int_a^b |K_M(s)|\, ds < \int_a^b |K_T(s)|\, ds \; .$$



(4) The history of convergence of the approximation error:

$$e_n := \int_0^{\pi} \sin x \, dx - \underbrace{\sum_{l=1}^{n} w_l \sin x_l}_{\text{Gauss-Legendre quadrature}}$$

is below. We assume that $e_n = c\, e^{-nr}$ to assess if the convergence is exponential.

n	e_n	r_n
2	6.92 E-02	–
3	1.39 E-03	3.83
4	1.58 E-05	4.48
5	1.10 E-07	4.97
6	5.23 E-10	5.35
7	1.79 E-12	5.68
8	4.44 E-15	6.01

The fact that the mapping $n \mapsto r_n$ is increasing and does not seem to converge to a limit as n grows, indicates that the convergence is <u>faster</u> than exponential.

In class, we obtained the estimate

$$|e_n| \leq \pi \, \| K^{(n)} \|_{L^\infty(0,\pi)} \, \left\| \frac{d^{(2n)}}{ds^{(2n)}} \sin x \right\|_{L^\infty(0,\pi)}$$

where the Peano kernel is

$$K^{(n)}(s) := \int_0^{\pi} \frac{(x-s)_+^{2n-1}}{(2n-1)!} - \sum_{l=1}^{n} \omega_l \frac{(x_l - s)^{2n-1}}{(2n-1)!} .$$

A crude estimate which takes into account that $\omega_l > 0$ and $\sum_{l=1}^{n} \omega_l = \pi$, gives

$$\| K^{(n)} \|_{L^\infty(0,\pi)} \leq \frac{\pi^{2n}}{(2n-1)!} + \pi \cdot \frac{\pi^{2n-1}}{(2n-1)!} = 2 \frac{\pi^{2n}}{(2n-1)!}$$

and so

$$|e_n| \leq \frac{2\pi^{2n+1}}{(2n-1)!}$$

Using Stirling approximation, $(2n-1)! \gtrsim \sqrt{2\pi}\,(2n-1)^{1/2} \left(\frac{2n-1}{e}\right)^{2n-1}$ gives

$$|e_n| \leq \frac{2\pi^{2n+1}}{\sqrt{2\pi}\sqrt{e}} \left(\frac{e}{2n-1}\right)^{2n-1/2}$$

$$= \left[\frac{2\pi^{3/2}}{\sqrt{2\pi e}}\right] \left(\frac{e\pi}{2n-1}\right)^{2n-1/2}$$

$$= \left[\frac{\sqrt{2}\,\pi}{e}\right] \left[\frac{e\pi}{2n-1}\right]^{2n-1/2} .$$

This gives a convergence which is *faster* than exponential, in agreement with our experiments. However, the rate of convergence $r_n \approx + 2\ln\left(\frac{e\pi}{2n-1}\right)$ is far slower than the one observed experimentally. To catch it, we need to get a better estimate of the error.