Homework #8: Galerkin orthogonal projections. Due on Friday, November 16.

Consider the problem of approximating the function $u(x)$ on the domain $(0,1)$ by using the L^2-projection into the space $W_n := \{w \in \mathcal{C}^0(0,1) : w|_{I_i} \in P_k(I_i)\ i = 1,\ldots,N\}$.

1. (5 points) For intervals I_i of the same size, h, and $k=1$, define a basis of W_n and obtain the general form of the matrix equation defining the degrees of freedom of $\pi_n u$. Display such equation for $N=5$ and $u(x)=\sin(\pi x)$. What is the expected rate of convergence? Implement the method and display a history of convergence of the approximation error. Do your results match the theory?

2. (5 points) For intervals I_i of the same size, $h=1/N$, and arbitrary $k \geq 1$, define a basis of W_n and obtain the global matrix equation defining the degrees of freedom of $\pi_n u$. For general $k>1$, find the local bases, the local matrices, and show how to assemble the global matrix equation. Display such equation for $N=5$, $u(x)=\sin(\pi x)$ and $k=3$.

Hint: To define a local basis for any $k \geq 1$, use the scaled Legendre polynomials so that all the integrals defining the global matrix can be computed by hand.

3. (5 points) For intervals I_i of the same size, $h=1/N$, and arbitrary $k \geq 1$, let us use the technique of static condensation to implement the method under consideration. First, show how to split the space W_n into its subspace of bubble functions W_n^b and its subspace W_n^∂. Then, find the weak formulation defining $(\pi_n u)^\partial$. Finally, find the local bases, the local matrices and show how to obtain the global matrix equation defining the degrees of freedom of $(\pi_n u)^\partial$. Display such equation for $N=5$, $u(x)=\sin(\pi x)$ and $k=3$.

4. Write a code that implements the method. For $k=3$, display a history of convergence of the approximation error as the number of intervals N is doubled. Do your results match the theory? (Use exact integration to evaluate the right-hand side. Use Gaussian quadrature to approximate the $L^2(0,1)$-norm of the error.)