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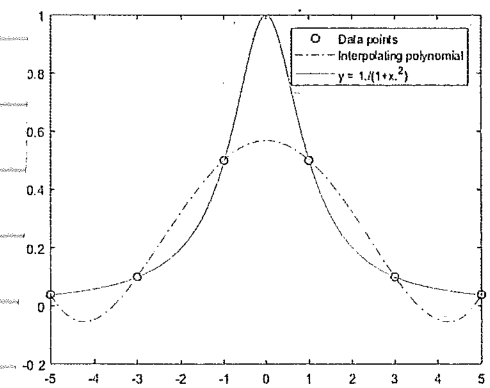
Homework #9: Interpolation and Filters

(1) We consider the problem of interpolating with polynomials of degree N the function $u(x) = (1+x^2)^{-1}$ in the interval $[-5,5]$ with equidistant points. So:

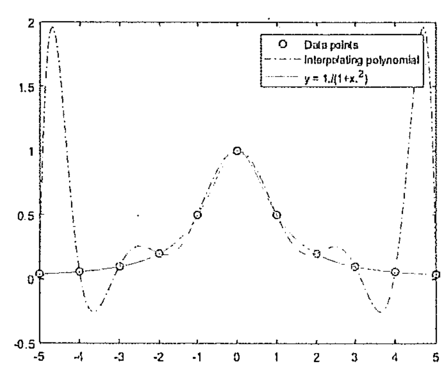
$$I_N u(x_j) = u(x_j) \qquad x_j = -5 + \frac{10}{N} j \ , \ j = 0, \dots, N.$$

In the figures, we plot $I_N u$ and u for $N=5$ (top) and $N=10$ (bottom). We see a better convergence of $I_N u$ to u for the points near to $x=0$ than for the points near to $|x|=5$.



N=5

In the Table of convergence below, we see convergence for $(I_N u - u)(x)$ for $x=3$ but not for $x=4$.

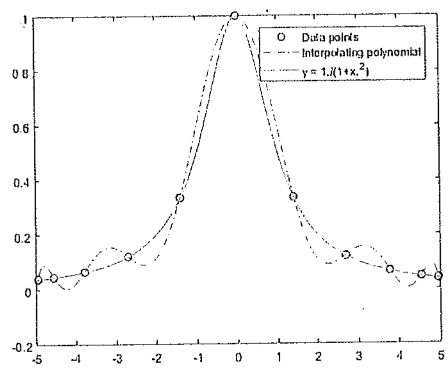


N=10

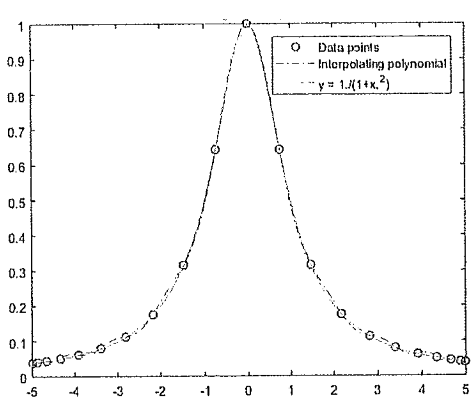
This is in full agreement with the results in the paper by James F. Epperson. in the American Math. Monthly (1987) pp. 329-341

N	$\lvert I_N u-u\rvert (x=3)$	α_N	$\lvert I_N u-u\rvert (x=5)$	α_N
4	$2.10 \; 10^{-1}$	—	$4.38 \; 10^{-1}$	—
16	$5.52 \; 10^{-2}$	0.96	$8.90 \; 10^{-1}$	-0.51
32	$1.75 \; 10^{-2}$	0.83	$1.76 \; 10^{0}$	-0.49
64	$4.18 \; 10^{-4}$	2.69	$2.99 \; 10^{1}$	-2.04
128	$1.31 \; 10^{-5}$	2.50	$4.23 \; 10^{3}$	-3.57

② When we us the Chebyshev points $x_j := 5\cos\frac{2j+1}{2N+2}\pi$, $j=0,\dots,N$, we see that we seem to have convergence in the whole interval



N = 10



N = 20

Cauchy's error estimate gives, for $|x| \leq 5$,

$$e(x) = \prod_{j=0}^{N} (x - x_j) \quad \frac{u^{(N+2)}(\xi)}{(N+2)!} \qquad \text{for some } \xi \in [-5, 5].$$

$$= 5^{N+1} \cdot \prod_{j=0}^{N} \left(\frac{x}{5} - \frac{x_j}{5} \right) \cdot \frac{u^{(N+2)}(\xi)}{(N+2)!}$$

$$\Rightarrow \quad |e(x)| \leq 5^{N+1} \, 2^{-N+1} \cdot \frac{|u^{(N+2)}(\xi)|}{(N+2)!}$$

$$= \left(\frac{5}{2}\right)^{N} \cdot 10 \; \frac{|u^{(N+2)}(\xi)|}{(N+2)!}$$

Since $|u^{(N+2)}(\xi)| \leq c_0 \, (N+2)!$, we get

$$|e(x)| \leq c_0 \cdot 10 \left(\frac{5}{2}\right)^{N}$$

This means that we cannot explain the convergence of $I_N u$ to u by using Cauchy's error estimate.

(3) We have that

$$S^F(\Pi_N u)(x) = \frac{1}{2\pi} \int_0^{2\pi} K_N^F(x-y) \, u(y) \, dy \, , \qquad K_N^F(z) = \frac{1}{N} \frac{\sin^2(Nz/2)}{\sin^2(z/2)}.$$

Since $u \geq 0$, $\quad S^F(\Pi_N u) \geq 0 \Rightarrow$ no undershoots

Since $u \leq 1$, $\quad S^F(\Pi_N u) \leq \frac{1}{2\pi} \int_0^{2\pi} K_N^F(z) \, dz = 1 \Rightarrow$ no overshoots.

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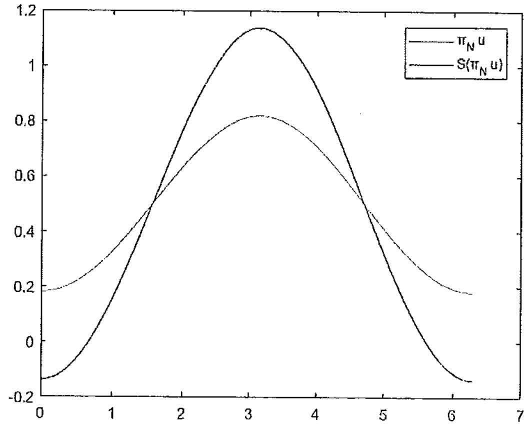
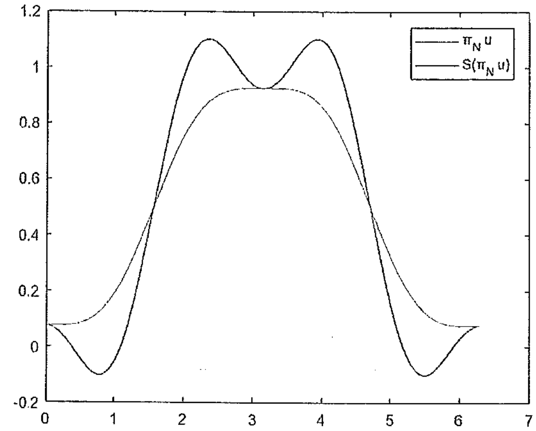
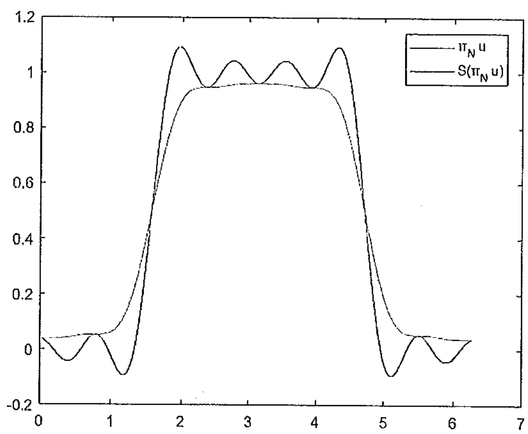
Féjer kernel



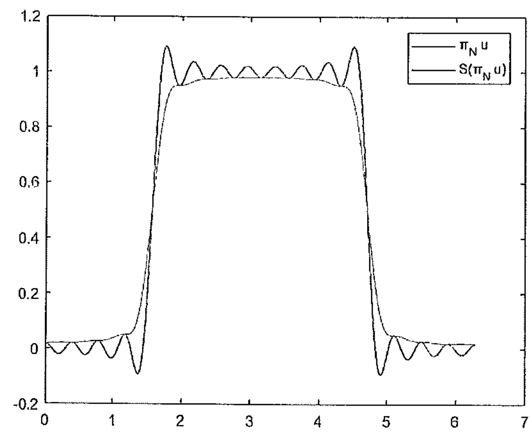
N=2



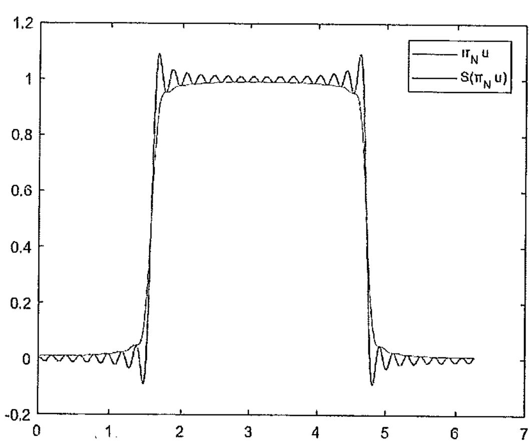
N=4



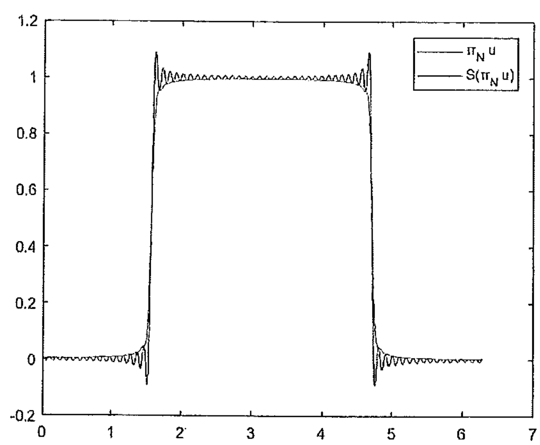
N=8



N=16



N=32



N=64

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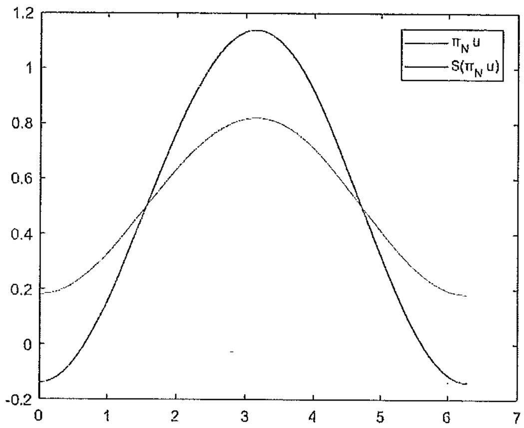
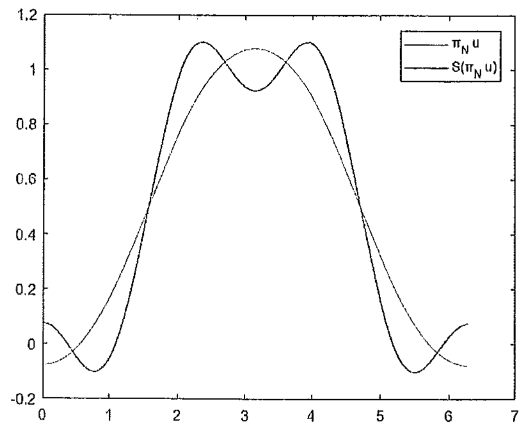
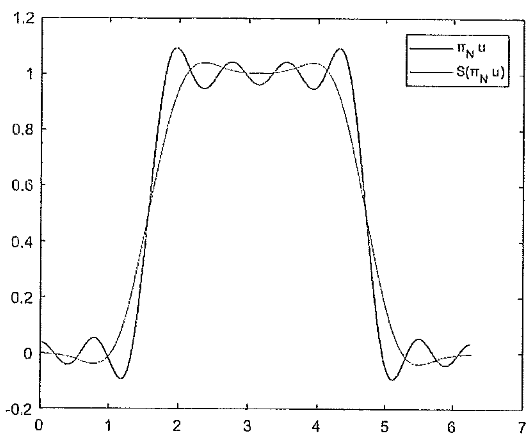
Vandeven filter with $p=4$



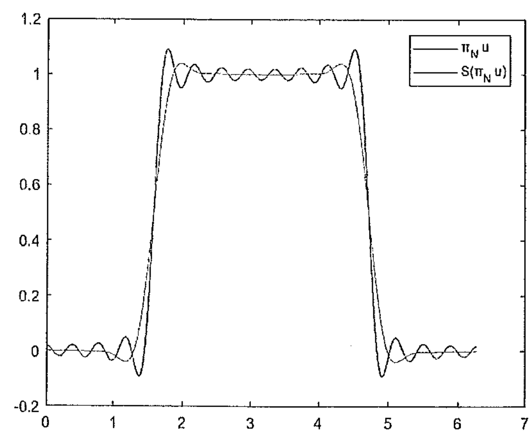
N=2



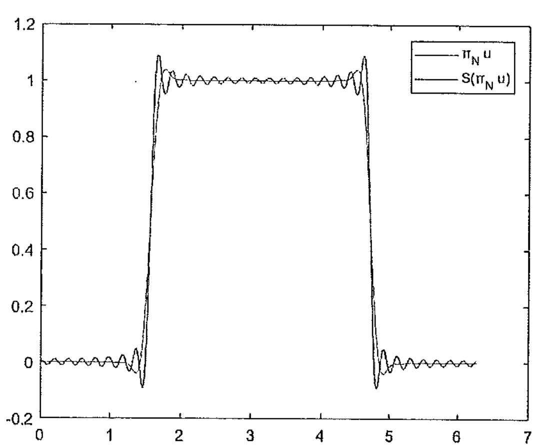
N=4



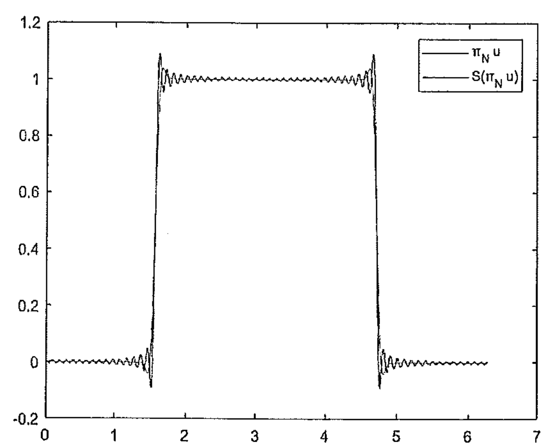
N=8



N=16



N=32



N=64

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For $x = \pi$, we have that $S_V(\Pi_N u)(x)$ does converge with order better than one, as would $\Pi_N u(x)$ converge.

N	e_N	α_N	C_N
4	$7.67\ 10^{-2}$	–	–
8	$2.40\ 10^{-3}$	5.00	78.3
16	$7.44\ 10^{-5}$	5.00	80.4
32	$2.32\ 10^{-6}$	5.00	78.3
64	$7.26\ 10^{-8}$	5.00	78.1