Homework #: Interpolation and Filters. Due on Wednesday, December 12, 2018.

In what follows, $I_N u$ is the polynomial interpolant of the function $u : (-5, 5) \mapsto \mathbb{R}$ at the points x_j, $j = 0, \dots, N$.

1. **Equidistant points**. Consider the function $u(x) = 1/(1 + x^2)$ and the interpolation points $x_j := -5 + \frac{10}{N} j$, $j = 0, \dots, N$. Plot the interpolant $I_N u$ for various values of N. What happens when N increases? Show numerical evidence that $I_N u(3)$ converges to $u(3)$ but that $I_N u(4)$ does not. Justify your results.

Hint: To justify your results, a reference to a full explanation of the phenomenon is enough.

2. **Chebyshev points**. Repeat the above exercise this time with the Chebyshev points $x_j := 5 \cos \pi \frac{2j+1}{2N+2}$ $j = 0, \dots, N$. Plot the interpolant $I_N u$ for various values of N. What happens when N increases. Do we have convergence on the whole interval [-5,5]?

3. **Féjer filter.** Consider the function

$$u(x) = \begin{cases} 1 & x \in (\pi/2, 3\,\pi/2), \\ 0 & \text{otherwise.} \end{cases}$$

For several values of N, compare in a figure (or figures) the approximations $\pi_N u$ and $S^F(\pi_N u)$, where S^F denotes the Féjer smoothing operator with kernel

$$K_N^F(x) := \sum_{k=-N}^{N} (1 - |k/N|)\, e^{ikx}.$$

Explain the absence of over- and under-shoots.

Hint: Use the fact that it can be proven that

$$K_N^F(y) = \frac{1}{N} \frac{\sin^2(Ny/2)}{\sin^2(y/2)} \geq 0.$$

4. **Vandeven filter.** Repeat the above exercise but this time use the Vandeven filter with $p = 4$. Display a history of convergence table for the absolute value of the error at $x = \pi$. We know that at that point, the order of convergence of $u - \pi_N u$ is of order $1/N$. Is that also the case for the Vandeven filter?