MATH 4567, SPRING 2019
HOMEWORK PROBLEMS No.2
Due on March 6

Problem 1. Find the best approximation g in the mean on the interval $0 \leq x \leq \pi$ for the function $f(x) = 1$ using linear combinations of

$$f_1(x) = \sin x, \quad f_2(x) = \sin 2x, \quad f_3(x) = \sin 3x.$$

Then evaluate the error of approximation, that is, $\|f - g\|$ in $L^2[0, \pi]$.

Problem 2. On the interval $[0, \pi]$ find

a) the Fourier sine series for the function f in Problem No. 2(b) on page 12;
b) the Fourier cosine series for the function f in Problem No. 4 on page 13;
c) the Fourier cosine series for the function f in Problem No. 3(a) on page 12.

In addition, in c) write down Parseval's equality corresponding to this Fouirer series and use it to evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Problem 3. Let $S_n f(x)$ denote the n-th partial sum of the Fourier series in $-\pi \leq x \leq \pi$ for the function defined to be $f(x) = x + 1$ for $x > 0$, $f(x) = 2x - 3$ for $x < 0$, and $f(0) = 0$.

a) Evaluate for each $x \in [-\pi, \pi]$ the limit $S(x) = \lim_{n\to\infty} S_n f(x)$.
b) Sketch the graph of S on the whole real line.
c) Find the values $S(10)$, $S(20)$.

Hint: The function S is 2π-periodic, so it is enough to know its values on $(-\pi, \pi]$.

Problem 4. Check that the function $f(x) = -\log x$ belongs to $L^2[0, 1]$ and find its L^2-norm. Then consider its cosine Fourier series in the interval $0 \leq x \leq 1$, that is,

$$-\log x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\pi n x).$$

Evaluate the sum $\sum_{n=1}^{\infty} a_n^2$ without computing the coefficients a_n.

Hint: Apply Parseval's equality for the cosine Fourier series in $[0, c]$. You will also need to compute a_0.