Problem 1. Given a continuous function \( f(x) \) on \([0, \pi]\), solve the temperature problem:
\[
\begin{align*}
    u_t &= ku_{xx}, \\
    u_x(0, t) &= u(0, t), \\
    u_x(\pi, t) &= u(\pi, t), \\
    u(x, 0) &= f(x).
\end{align*}
\]

Write your answer in the form of an infinite functional series
\[
    u(x, t) = \sum_{n=0}^{\infty} c_n y_n(x) T_n(t).
\]

Describe the functions \( y_n(x) \) and \( T_n(t) \) that are involved and indicate how to compute the coefficients \( c_n \) in terms of \( f \).

Note. You will need to consider an associated Sturm-Liouville problem which is exactly the same as the one in Problem 4 from HW3 with parameters \( \beta = 1 \) and \( c = \pi \).

Problem 2. Given parameters \( A, B, C \) (real), consider the temperature problem with non-homogeneous boundary conditions:
\[
\begin{align*}
    u_t &= ku_{xx}, \\
    u_x(0, t) &= u(0, t) + A, \\
    u_x(\pi, t) &= u(\pi, t) + B, \\
    u(x, 0) &= Cx.
\end{align*}
\]

Reduce it to Problem 1 by virtue of a suitable substitution \( u(x, t) = U(x, t) + \Phi(x) \). Indicate new initial temperatures \( F(x) \) in the homogeneous problem about \( U(x, t) \).

Problem 3. Use the Fourier transform to solve the temperature problem in the upper half-plane
\[
\begin{align*}
    u_t &= ku_{xx}, \\
    u(x, 0) &= e^{-2x^2}.
\end{align*}
\]

Hint. First formulate a general theorem about the boundary value problems
\[
\begin{align*}
    u_t &= ku_{xx}, \\
    u(x, 0) &= f(x).
\end{align*}
\]

Problem 4. Use the Fourier transform to solve the boundary value problem, involving the Laplace equation:
\[
\begin{align*}
    \Delta u &= 0, \\
    u_x(0, y) &= 0, \\
    u(x, 0) &= \frac{1}{1+x^2}.
\end{align*}
\]
Hint. First formulate a general theorem about boundary value problems of the form

\[ \Delta u = 0, \quad u = u(x, y), \quad x, y \geq 0, \quad u \text{ is bounded}, \]
\[ u_x(0, y) = 0, \]
\[ u(x, 0) = f(x). \]

Note that the Fourier transforms for the functions

\[ f(x) = e^{-x^2/(2\sigma^2)} \quad \text{and} \quad f(x) = \frac{1}{1 + x^2} \]

(needed in Problems 3 and 4) are known and have been evaluated in class.