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# COMPACT SETS OF ADDITIVE MEASURES


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In this note one gives compactifications, having a general character, for the family of all probability measures and for convex, bounded sets in linear spaces, considered in weak topologies.


In this note we present an approach, based on the application of the technique of additive measure, allowing us to cover from one single point of view several apparently heterogeneous facts such as Yu. V. Prokhorov's theorem on the compactness of dense families of probability measures, reflexivity criteria for normed spaces, the theorem on the weak compactness of a ball in the conjugate space, the existence of the Cech compactification, and also to obtain other, less well-known statements.

Let $C$ be a family of bounded functions on a set $X$, let $\mathfrak{A}$ be an arbitrary $\sigma$-algebra in $X$ with respect to which the functions from $C$ are measurable. We denote by $\tilde{Z}(X)$ the vector space of finitely additive measures (not necessarily positive) defined on $\mathfrak{A}$ and having a bounded variation.

If $\{\mu_i\}$ is a net in $\tilde{Z}(X)$ then the formula $\mu_i \to \mu \Leftrightarrow \forall f \in C \ \int f\, d\mu_i \to \int f\, d\mu$ defines in $\tilde{Z}(X)$ a locally convex topology $w_C$ known as the topology of $C$-weak convergence. $\tilde{Z}_C(X)$ denotes $\tilde{Z}(X)$ with the topology $w_C$.

We mention the simplest properties of the $C$-weak topologies:

1) $w_C = w_{\operatorname{lin} C}$.
2) $C_1 \subset C_2 \Rightarrow w_{C_1}$ is weaker than $w_{C_2}$.
3) If the set $A$ in $\tilde{Z}(X)$ is bounded in variation, then $w_C|_A = w_{\operatorname{cl} C}|_A$.

We consider the canonical imbedding $X \subset \tilde{Z}_C(X)$: $X \ni x \longmapsto \delta_x \in \tilde{Z}_C(X)$, where $\delta_x$ is the Dirac measure at the point x. It is easy to see that the additive uniform structure inherited from $\tilde{Z}_C(X)$ is the smallest of those in which the functions from $C$ are uniformly continuous.

We mention one more simple property:

4) a) $\overline{\operatorname{span} X} = \tilde{Z}_C(X)$, i.e., $X$ is a fundamental set.
   b) The convex hull co $X$ is dense in $\tilde{D}_C(X) = \{\mu \in \tilde{Z}_C(X) : \mu \geq 0,\ \mu X = 1\}$.

In general, the space $\tilde{Z}_C(X)$ is not separated and therefore it is not convenient for inversions. Let $p_C : \tilde{Z}_C(X) \to Z_C(X) = \tilde{Z}_C(X)/L_0$ be the canonical projection, where $L_0$ is the inter-


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section of the neighborhoods of zero in $\tilde{Z}_C(X)$ . Actually, the factorization occurs with respect to the equivalence relation $\mu\sim\nu \Leftrightarrow \forall f\in C\ \int f d\mu=\int f d\nu$.

We denote $A_C = p_C(A)$ where $A\subset \tilde{Z}_C(X)$ , $\chi=\{1_B : B\in\mathfrak{A}\}$ . In the sequel, the following statement will be of a fundamental importance. We denote $\tilde{D}'_C(X)=\{\mu\in\tilde{Z}_C(X) : \operatorname{var}\mu\le 1\}$.

<u>Proposition.</u> Assume that the set $A\subset\tilde{Z}(X)$ is bounded in variation. If $A$ is closed in $Z_\chi(X)$ , then $A_C$ is compact.

<u>Proof.</u> By assumption $\alpha=\sup\{\operatorname{var}\mu : \mu\in A\}<\infty$ . For the sake of simplicity, assume that $\alpha=1$ . The topology of $\tilde{D}'_\chi(X)$ is induced from the compact space $[-1,1]^{\mathfrak{A}}$ , and $\tilde{D}'_\chi(X)$ itself is closed there, i.e., compact. We have: $A$ is closed in $Z_\chi(X) \Longrightarrow A$ is closed in $\tilde{D}'_\chi(X)\Rightarrow w_\chi|_A$ is a compact topology.

We denote by $I(X)$ the space of all measurable bounded functions on $X$ . Then $\operatorname{span}\chi=I(X)\Rightarrow w_\chi|_A=w_{I(X)}|_A \geqslant w_C|_A\Rightarrow w_C|_A$ is a compact topology. The projection $p_C$ is continuous $\Rightarrow A_C=p_C(A)$ is compact.

As applications of the above-proved proposition, we give four corollaries.

<u>COROLLARY 1.</u> Let $L$ be a vector space, let $L'$ be its algebraic conjugate, let $C$ be a linear subspace of $L'$ , let $\sigma(L,C)$ be the weakest topology with respect to which the functionals from $C$ are continuous. Let $S$ be a convex set, closed and bounded in the $\sigma(L,C)$ topology of the space $L$ . In the linear space $C_S=\{A|_S : A\in C\}$ we consider the standard norm $\|A|_S\|=\sup_{x\in S}|Ax|$ . The following three statements are equivalent:

1) $S$ is compact.

2) Any continuous linear functional $\varphi\in C_S^*$ has the form $\varphi(A|_S)=Ax$ for some $x\in \operatorname{lin} S$ .

3) $S$ is complete (in the additive structure of $(L,\sigma(L,C))$ ).

For the proof it is sufficient to note that:

a) If $\varphi\in C_S^*$ then there exists $\mu\in\tilde{Z}(S)$ with the property $\varphi(A|_S)=\int_S A\,d\mu$ . The finitely additive measures of bounded variation have the form $\alpha_1\mu_1-\alpha_2\mu_2$ , where $\mu_1,\mu_2\in\tilde{D}(S)$, $\alpha_1,\alpha_2\geqslant 0$ .

b) Since the functionals from $C$ are linear and $S$ is convex, it follows that $\operatorname{co} S_{C_S}=S_{C_S}$ and is everywhere dense in $D_{C_S}(S)=[\tilde{D}_{C_S}(S)]_{C_S}$.

c) $S$ is compact (complete) $\Leftrightarrow S_{C_S}$ is compact (complete).

d) By the previous proposition, $D_{C_S}(S)$ is a separated compact space and, consequently, $S_{C_S}$ is compact $\Leftrightarrow S_{C_S}=D_{C_S}(S) \Leftrightarrow$ for each probability finitely additive measure $\mu$ there exists $x\in S$ such that $\int_S A\,d\mu=Ax$ for any $A\in C$ .

Particular cases of the corollary are:

1) $X$ is a normed space. Then $X$ is reflexive $\Leftrightarrow$ the unit ball $B_X$ is weakly compact. One has to set: $L=X$ , $S=B_X$, $C=L^*$.

2) $X$ is a normed space $\Rightarrow B_{X^*}$ is $\sigma(X^*,X)$ -compact. One has to set $L=X^*$ , $S=B_{X^*}$ , $C=\{j_x : x\in X\}$ where $j_x(A)=Ax$ for all $A\in X^*$.

<u>COROLLARY 2.</u> Let $X$ be a completely regular topological space and let $C=C(X)$ be the space of continuous bounded functions. Then $\operatorname{cl}_{D_C(X)}X$ is the Cech compactification.

<u>Proof.</u> The continuous functions from $C(X)$ are uniformly continuous in the uniform structure $X$ by virtue of the above made remark on the trace of the structure $\tilde{Z}_C(X)$ on $X$ .

The following interesting corollary, from which we obtain at once the nonreflexivity of $C(K)$ for infinite compacta $K$, shows that the weak compact sets in $C(K)$ are in a certain sense "rarefied." For the sake of simplicity we shall asumme that $S \subset B_{C(K)}$.

COROLLARY 3. If $S$ is weakly compact, then $\sup_{f\in S}|\int f d\mu| < \operatorname{var}\mu$ for some regular Borel measure $\mu$.

Proof. From the assumption it follows that $S_0 = \operatorname{cl}\operatorname{co} S$ is a convex, weakly compact set and, since $S \subset S_0$, assuming the opposite of the statement, we obtain the relation $\sup_{f\in S_0}|\int f d\mu| = \operatorname{var}\mu$ for all the regular measures. If in $Z_{S_0}(K)$ we introduce the norm $\|\mu\| = \sup_{f\in S_0}|\int f d\mu|$ then it will be isometric to the space of all regular measures $Z_{reg}$ with the norm var. By Corollary 1, for any $A \in Z^*_{reg}$ there exists $f_0 \in \operatorname{lin} S_0$ such that $A\mu = \int f_0 d\mu$ for all measures from $Z_{reg}$. We consider the functional $A\mu = \mu\{x_0\}$ where $x_0 \in K$, $\mu \in Z_{reg}$. $|A\mu| \leqslant \operatorname{var}\mu \Rightarrow A \in Z^*_{reg} \Rightarrow \exists f_0 \in C(K)\ \int f_0 d\mu = \mu\{x_0\}$. Since the Dirac measures $\delta_x$ are regular, for all $x \in K$ $f_0(x) = \delta_x\{x_0\} \Rightarrow \{x_0\}$, is open in $K \Rightarrow K$ is discrete $\Rightarrow K$ is finite?!

Finally, we give one more corollary, which is a direct theorem of Yu. V. Prokhorov. Let $S$ be a normal topological space, let $Z_0$ be a set of Borel probability measures with topology induced from $Z_{C(S)}(S)$. We recall that a set $A \subset Z_0$ is said to be dense if for each $\varepsilon > 0$ there exists a compactum $K \subset S$ such that $\forall\mu \in A$ $\mu K > 1-\varepsilon$.

COROLLARY 4. The dense families in $Z_0$ are relatively compact.

Proof. We shall simply write $D = D_{C(S)}(S)$ and we shall assume that the distinct measures in $Z_0$ are nonequivalent, i.e., $Z_{0\,C(S)} = Z_0$; otherwise, we would consider the quotient space. If $A$ is a dense family of measures, then there exists an increasing sequence of compacta $\{K_n\}$ for which $\mu K_n \geqslant 1 - 1/2^n$ for any $\mu \in A$. $A$ is a relatively compact in $Z_0 \Leftrightarrow \operatorname{cl}_{Z_0} A$ is compact. But $\operatorname{cl}_{Z_0} A = \operatorname{cl}_D A \cap Z_0$ is compact $\Leftrightarrow \operatorname{cl}_D A \cap Z_0$ is closed since $D$ is a separated compact space. Let $\{\mu_i\}$ be a directed family of measures in $\operatorname{cl}_D A \cap Z_0$ and assume that $\mu_i \to \mu \in D$. Since $\mu_i \in \operatorname{cl}_D A$, we also have $\mu \in \operatorname{cl}_D A$. Therefore, it is sufficient to show that $\mu$ is equivalent to some probability measure. Since $\forall\tilde\mu \in A$ $\tilde\mu K_n \geqslant 1 - 1/2^n$, by A. D. Aleksandrov's theorem [1], there exists a finitely additive probability measure $\nu$, defined on the algebra of sets containing all the open sets, equivalent to μ, regular, and, consequently, with the property $\nu K_n \geqslant 1 - 1/2^n$, $n \in \mathbb{N}$. By the Riesz–Markov theorem, there exists a countably additive regular measure $K_n$ on $\nu_n$, such that $0 \leqslant \nu_n \leqslant 1$ and $\forall f \in C(S)\ \int_{K_n} f d\mu = \int_{K_n} f d\nu_n$. We extend $\nu_n$ by zero outside $K_n$. Since $\nu_n$ is regular, we have $\nu_n \geqslant 1 - 1/2^n$ and, consequently, $\forall f \in C(S)\ |\int f d\nu_{n+1} - \int f d\nu_n| \leqslant \frac{\sup|f|}{2^{n-1}}$. From the normality of $S$ it follows that for each Borel set $B$ we have $|\nu_{n+1}B - \nu_n B| \leqslant 1/2^{n-1}$. Therefore, $\operatorname{var}(\nu_{n+1} - \nu_n) \leqslant 2\sup_B|\nu_{n+1}B - \nu_n B| \leqslant 1/2^{n-2}$. Thus, the sequence of measures $\{\nu_n\}$ is fundamental (i.e., it is a Cauchy sequence) and by virtue of the completeness of the metric $\rho(\mu,\nu) = \operatorname{var}(\mu-\nu)$, in the space of positive finite measures (countably additive) it has a limit $\lambda$. Then, $\nu_n \xrightarrow[\operatorname{var}]{} \lambda \Rightarrow \nu_n S \to \lambda S \Rightarrow \lambda S = 1$. We show that $\lambda$ is the desired probability, i.e., $\forall f \in C(S)\ \int f d\lambda = \int f d\mu$:

$$|\int f d\mu - \int f d\lambda| = |\int f d\nu - \int f d\lambda| \leqslant |\int_{K_n} f d\nu - \int_{K_n} f d\nu_n| + |\int_{K_n} f d\nu_n - \int_{K_n} f d\lambda| + |\int_{S\setminus K_n} f d\nu - \int_{S\setminus K_n} f d\nu_n| + |\int_{S\setminus K_n} f d\nu_n - \int_{S\setminus K_n} f d\lambda| \leqslant$$

$$\leqslant \|f\|\left(2\operatorname{var}(\nu_n - \lambda) + \frac{3}{2^n}\right) \xrightarrow[n\to\infty]{} 0 \quad \Rightarrow \int f d\mu = \int f d\lambda.$$

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# DISTRIBUTION OF INTEGRAL FUNCTIONALS OF A BROWNIAN MOTION PROCESS

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In this paper one considers methods which enable one to determine the distribution of certain functionals of a Brownian motion process. Among such functionals we have: the positive continuous additive functional of a Brownian motion, defined by the formula

$$A(t)=\int_{-\infty}^{\infty}\hat{t}(t,y)\,dF(y),$$

where $\hat{t}(t,y)$ is the Brownian local time process while $F(y)$ is a monotonically increasing right continuous function; the functional

$$B(t)=\int_{-\infty}^{\infty}f(y,\hat{t}(t,y))\,dy,$$

where $f(y,x)$ is a continuous function; and the functional

$$C(t)=\int_0^t f(w(s),\hat{t}(s,z))\,ds$$

As an application of these methods one considers some concrete functionals such that $\hat{t}^{-1}(z)=\min\{s:\hat{t}(s,0)=z\}, \int_{-\infty}^{\infty}\hat{t}^2(t,y)\,dy$, $\sup_{y\in R^1}\hat{t}(T,y)$, where $T$ is an exponential random time, independent of $\hat{t}(t,y)$.

0. In this paper we consider methods which allow us to determine the distribution of certain functionals of the process of Brownian motion. The first class contains additive functionals of the Brownian motion. We denote by $w(s)$ the standard process of Brownian motion on a line. The limit (see [1])

$$\hat{t}(t,y)=\lim_{\varepsilon\to 0}\varepsilon^{-1}\int_0^t \mathbb{1}_{[y,y+\varepsilon)}(w(s))\,ds,$$

where $\mathbb{1}_A$ is the indicator of the set A, exists with probability one and is called the Brownian local time at the point $y$ over the time $t$. In a certain sense, the Brownian local time is an elementary additive functional of the Brownian process. Thus, any positive continuous additive functional $A(t)$ of the Brownian motion can be represented (see [2]) in the form of the mixture

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