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# VARIATIONS OF RANDOM PROCESSES WITH INDEPENDENT INCREMENTS

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In the paper one considers random processes $\{\xi_s\}_{0\le s\le t}$ with independent increments, continuous in the mean $(\forall p<\infty)$. One establishes relations among multiple integrals, variations, i.e., the limits of sums of the form $\sum(\xi_{t_i}-\xi_{t_{i-1}})^n$, and the Itô stochastic integrals.

## 0. Notations and Definitions

Let $(\Omega,\mathfrak{F},P)$ be a probability space, let $(\mathfrak{X},\mathfrak{A})$ be a measurable space, and let $\mathfrak{P}$ be a semiring of sets that generate $\mathfrak{A}$. By a process (or measure) with independent increments we shall mean a mapping $\mu:\mathfrak{P}\to L^0(\Omega,\mathfrak{F},P)$ satisfying the conditions:

a) $\mu$ is additive, i.e., $\forall A,B\in\mathfrak{P}$ $A\cap B=\emptyset\Rightarrow\mu(A\cup B)=\mu A+\mu B$,

b) for any finite collection of nonintersecting sets $A_1,\ldots,A_N$ from $\mathfrak{P}$, $\mu A_1,\ldots,\mu A_N$ are independent random variables.

A process $\mu$ is said to be $m$-continuous if $m$ is a finite, positive, continuous measure defined on the $\sigma$-algebra $\mathfrak{A}$ and if for some sequence $\alpha_n\geq 0$ we have $|E(\mu A)^n|\leq\alpha_n mA$ for each $A$ from $\mathfrak{P}$. A process $\mu$ is said to be strongly continuous if for some $m$ it is $m$-continuous.

## 1. Extension of Processes

Let $\mu:\mathfrak{P}\to L^0(\Omega,\mathfrak{F},P)$ be a process with independent increments. We denote $Z(\mu)=\{m:\mu$ is $m$-continuous$\}$. Clearly, the condition of strong continuity means that $Z(\mu)\neq\emptyset$. We introduce on $Z(\mu)$ an order structure: $m_1\leq m_2\Leftrightarrow\forall A\in\mathfrak{A}$ $m_1A\leq m_2A$. Running slightly ahead, we mention that many properties and the definition of $m$-continuous processes, in which $m$ occurs and which will be considered below, actually do not depend on $m$, provided the process $\mu$ is strongly continuous. This circumstance explains in a great deal

THEOREM 1. If the process $\mu$ is strongly continuous, then $Z(\mu)$ is a lattice.

Proof. First we mention that an ordered set $Z$ is said to be a lattice if $\forall x,y\in Z$ $\exists x\wedge y=\inf\{x,y\},x\vee y=\sup\{x,y\}$. We denote by $Z$ the family of all finite measures on the measur-

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able space $(\mathfrak{X},\mathfrak{A})$. We introduce on $Z$ the same ordering as on $Z(\mu)\subset Z$. From Hahn's theorem on the decomposition of a finite measure there follows that $Z$ is a lattice.

Indeed, let $m_1$ and $m_2\in Z$, $m=m_1-m_2$. Then there exist $A,B\in\mathfrak{A}$ such that $A\cap B=\emptyset$, $A\cup B=\mathfrak{X}$, $m|_A\geq 0$, $m|_B\leq 0$. We set $m'C=m_2(A\cap C)+m_1(B\cap C)$, $m''C=m_1(A\cap C)+m_2(B\cap C)$. Obviously, $m'=m_1\wedge m_2$, $m''=m_1\vee m_2$. Moreover, if $m_1$ and $m_2$ are continuous, then also $m', m''$ are continuous measures. Since the order in $Z(\mu)$ is inherited from $Z$, it is sufficient to show that $\forall m_1, m_2\in Z(\mu)$ $m_1\wedge m_2, m_1\vee m_2\in Z(\mu)$. We recall that $m\in Z(\mu)\Leftrightarrow$ $m$ is a finite, positive, continuous measure and for some sequence $\alpha_n\geq 0$ we have $|E(\mu A)^n|\leq\alpha_n mA$. From the last inequality it can be seen that if $m, m'$ are continuous measures, than $m\in Z(\mu)$ implies $m'\in Z(\mu)$, as soon as $m\leq m'$. Since $m_1\leq m_1\vee m_2$ and $m_1\in Z(\mu)$, then, consequently, $m_1\vee m_2\in Z(\mu)$.

Let $A, B$ be those measurable sets which have been mentioned above for the measure $m=m_1-m_2$; $|E(\mu C)^n|\leq\alpha_n m_1 C$, $|E(\mu C)^n|\leq\beta_n m_2 C \Longrightarrow |E(\mu C)^n|=\left|\sum_{k=0}^{n} E[\mu(A\cap C)]^k\cdot E[\mu(B\cap C)]^k\cdot C_n^k\right|\leq\sum_{k=0}^{n}C_n^k\beta_k\alpha_{n-k}\cdot m_2(A\cap C)\cdot m_1(A\cap C)\leq$ $\leq\left(\sum_{k=0}^{n}\frac{1}{2}C_n^k\cdot\alpha_{n-k}\cdot\beta_k\right)\cdot[m_1\wedge m_2(C)]^2\leq\gamma_n\cdot m_1\wedge m_2(C)$, where $\gamma_n=\frac{m_1\wedge m_2(\mathfrak{X})}{2}\sum_{k=0}^{n}C_n^k\alpha_{n-k}\cdot\beta_k$. Thus, $m_1\wedge m_2\in Z(\mu)$.

<u>COROLLARY.</u> We denote $\mathcal{U}_m(\varepsilon)=\{\tau: \mathrm{rank}\,\tau(m)<\varepsilon\}$, where $\varepsilon>0$, $m\in Z(\mu)$, $\tau=\{A_1,\dots,A_N\}$ is a partition of $\mathfrak{X}$, $A_i\in\mathfrak{A}$, $\mathrm{rank}\,\tau(m)=\max mA_i$. The family $\{\mathcal{U}_m(\varepsilon):\varepsilon>0\}$ forms a basis of the filter $\mathcal{F}_m$, denoted usually as $\mathrm{rank}\,\tau(m)\longrightarrow 0$. From Theorem 1 it follows that for a strongly continuous process $\mu$, $\mathcal{F}(\mu)=\bigcup_{m\in Z(\mu)}\mathcal{F}_m$ is a filter.

We consider now the question of the extension. As before, $\mathcal{P}$ is a semiring generating $\mathfrak{A}$ and $\mu:\mathcal{P}\longrightarrow L^0(\Omega,\mathcal{F},P)$ is a process with independent increments.

<u>THEOREM 2.</u> If the process $\mu$ is strongly continuous, then on the measurable space $(\mathfrak{X},\mathfrak{A})$ there exists a unique strongly continuous process $\mu^*$ with independent increments such that $\mu^*|_{\mathcal{P}}=\mu$. If $\mu$ is $m$-continuous, then also $\mu^*$ is $m$-continuous, i.e., $Z(\mu^*)=Z(\mu)$.

<u>Proof.</u> Let $f=\sum_{j=1}^{N}a_j 1_{A_j}$ be a step function based on $A_j$ from $\mathcal{P}$, and let $\mathcal{L}$ be the vector space of these functions. We set

$$I(f)=\sum_{j=1}^{N}a_j\,\mu A_j.$$

We note that one has the following algebraic equality:

$$\left(\sum_{j=1}^{N}x_j\right)^n=\sum_{\substack{\alpha_1,\dots,\alpha_s\geq 1\\ \sum\alpha_i=n\\ 1\leq s\leq n}}\frac{n!}{\alpha_1!\dots\alpha_s!\,s!}\cdot\sum_{\substack{i_p\neq i_q\\ (p\neq q)}}x_{i_1}^{\alpha_1}\cdot\ldots\cdot x_{i_s}^{\alpha_s},$$

where $n\in\mathbb{N}$, $x_1,\dots,x_N\in\mathbb{R}$.

Let $m\in Z(\mu)$, $|E(\mu A)^n|\leq\gamma_n mA$ for each $A$ from $\mathcal{P}$. If we set $x_j=a_j\cdot\mu A_j$, then for the interior sums we have

$$\left|E\sum x_{i_1}^{\alpha_1}\cdot\ldots\cdot x_{i_s}^{\alpha_s}\right|=\left|\sum(Ex_{i_1}^{\alpha_1})\cdot\ldots\cdot(Ex_{i_s}^{\alpha_s})\right|\leq\sum_{i_p\neq i_q,(p\neq q)}|a_{i_1}|^{\alpha_1}\cdot\ldots\cdot|a_{i_s}|^{\alpha_s}\gamma_{\alpha_1}\cdot\ldots\cdot\gamma_{\alpha_s}mA_{i_1}\cdot\ldots\cdot mA_{i_s}$$
$$\leq(\max_{1\leq i\leq n}\gamma_i)^s\int|f|^{\alpha_1}dm\cdot\ldots\cdot\int|f|^{\alpha_s}dm\leq c_s\int|f|^n dm$$

by Hölder's inequality for some constant $C_s > 0$. Thus,

$$|E\, I(f)^n| \leqslant \gamma_n' \cdot \int |f|^n dm, \text{ where } \gamma_n' = \sum \frac{n!}{\alpha_1! \cdots \alpha_s! S!} C_s .$$

Entending by continuity the linear mapping $I$ from the space $\mathcal{L}$ to $\mathcal{L}^2(\mathfrak{X}, \mathfrak{A}, m)$, we obtain the mapping $I^*: \mathcal{L}^2(\mathfrak{X}, \mathfrak{A}, m) \to L^2(\Omega, \mathfrak{F}, P)$, which is continuous and satisfies the inequality $|E I^*(f)^n| \leqslant \gamma_n' \int |f|^n dm$, where $f \in \mathcal{L}^n(m)$ if $n$ is even and $f \in \mathcal{L}^{n+1}(m)$ if $n$ is odd. It remains for $A \in \mathfrak{A}$ to set $\mu^* A = I^*(1_A)$. Obviously, $\mu^*$ is a process with independent increments and $|E(\mu^* A)^n| \leqslant \gamma_n' \cdot mA$. At the same time we have established that $m \in Z(\mu^*)$. The uniqueness of the extension is obvious.

## 2. A Condition of Strong Continuity for Stochastic Continuous Processes with Independent Increments

Let $\mathfrak{X} = [a, b]$, let $\mathfrak{A}$ be the $\sigma$-algebra of Borel subsets of $[a, b]$. Let $\mathfrak{P}$ be the semiring of cells $[t, s), a \leqslant t \leqslant s \leqslant b$. If we have a process $\xi(t)$, $a \leqslant t \leqslant b$, with independent increments, such that $\xi(a) = 0$, then a measure $\mu[t, s) = \xi(s) - \xi(t)$ is connected with it, which, in accordance with our first definition is a process with independent increments. We also have the reverse relation: $\xi(t) = \mu[a, t)$.

When saying that a process $\xi(t)$ is strongly continuous, we apply this term to the process $\mu$. Obviously, the condition of the strong continuity of $\xi(t)$ is equivalent to the fact that for some nondecreasing function $F$, continuous on $[a, b]$, and a sequence $\alpha_n \geqslant 0$ one has

$$|E(\xi(s) - \xi(t))^n| \leqslant \alpha_n (F(s) - F(t)), \quad t < s.$$

Here we present, in terms of the Levy–Khinchin representation, the conditions that are necessary and sufficient for strong continuity. Let $\xi(t)$ be a stochastic continuous process with independent increments such that $\xi(a) = 0$. It is known [3] that there exist a continuous function $\gamma(t)$, a nondecreasing continuous function $D(t)$, and a function $G(t, x)$, continuous with respect to $t$ and nondecreasing with respect to $x \in \mathbb{R}$ such that $G(a, x) = 0$ and for $t < s$ the function $G(s, x) - G(t, x)$ does not decrease with respect to $x$ such that for $t < s$ one has

$$E \exp(i\lambda(\xi(s) - \xi(t)) = \exp\left\{ i\lambda(\gamma(s) - \gamma(t)) - \frac{\lambda^2}{2}(D(s) - D(t)) + \int \left(e^{i\lambda x} - 1 - \frac{i\lambda x}{1 + x^2}\right) \frac{1 + x^2}{x^2} (G(s, dx) - G(t, dx)) \right\}.$$

THEOREM 3. In order that the process $\xi(t)$ be strongly continuous it is necessary and sufficient that 1) the function $\gamma$ should have bounded variation and 2) $\int |x|^n G(b, dx) < \infty$ for all $n \in \mathbb{N}$.

Proof. Sufficiency. From condition 2) it follows that the random variable $\xi(s) - \xi(t), t < s$ has a characteristic function $f_{t,s}(\lambda)$, infinitely differentiable on the entire line and, consequently, also moments $a_n(t, s)$ of all orders [4], which are expressed in terms of the commutants $\varkappa_n(t, s)$ of the random variable $\xi(s) - \xi(t)$ according to the formula [4]:

$$a_n(t, s) = \sum \frac{n!}{i_1! (k_1!)^{i_1} \cdots i_\nu! (k_\nu!)^{i_\nu}} \cdot \varkappa_{k_1}(t, s)^{i_1} \cdots \varkappa_{k_\nu}(t, s)^{i_\nu},$$

where the summation is carried out over all collections $(i_1,\ldots,i_\nu;\,k_1,\ldots,k_\nu)$ of nonnegative integers, subjected to the condition $i_1k_1+\ldots+i_\nu k_\nu = n$. Therefore, it is sufficient to find a function $F$, nondecreasing and continuous on $[a,b]$ and a sequence $\alpha_n \geqslant 0$, such that for $t<s$ one should have $|\varkappa_n(t,s)| \leqslant \alpha_n(F(s)-F(t))$, and for this it is sufficient to find for each $n$ a function $[a,b]$ nondecreasing and continuous on $F_n$, such that $|\varkappa_n(t,s)| \leqslant F_n(s)-F_n(t)$ and then the function $F(t)=\sum\limits_{n=1}^{\infty}\frac{F_n(t)-F_n(a)}{(1+F_n(b)-F_n(a))\cdot 2^n}$ will be the desired one.

We write down the second characteristic of the random variable $\xi(s)-\xi(t)$. As it is known, $\varkappa_n(t,s)=i^{-n}\varphi^{(n)}_{t,s}(0)$. We have

$$\varkappa_1(t,s)=\gamma(s)-\gamma(t)+\int x\,(G(s,dx)-G(t,dx)),$$
$$\varkappa_2(t,s)=D(s)-D(t)+\int(1+x^2)(G(s,dx)-G(t,dx)),$$
$$x_n(t,s)=\int x^{n-2}(1+x^2)(G(s,dx)-G(t,dx)),\quad n>2.$$

We note that if $\varphi\in L^1(G(b,dx))$, then function $G_\varphi(t)=\int\varphi(x)G(t,dx)$ is continuous on $[a,b]$. For $\varphi_n(x)=|x|^n$ we denote $G_n=G_{\varphi_n}$. Then it is sufficient to set

$$F_1(t)=\operatorname{Var}\gamma\big|_a^t+G_1(t),$$
$$F_2(t)=D(t)+G_0(t)+G_2(t)$$
$$F_n(t)=G_{n-2}(t)+G_n(t),\quad n>2.$$

Necessity. From the strong continuity there follows the existence of all the moments $\xi(s)-\xi(t)$ and, consequently, the infinite differentiability at zero and of its second characteristic $\varphi_{t,s}(\lambda)$.

Therefore, the function $f(\lambda)=\int\left(e^{i\lambda x}-1-\frac{i\lambda x}{1+x^2}\right)\cdot\frac{1+x^2}{x^2}G(b,dx)$ has at zero all the derivatives. From the finiteness of the second moment of $\xi(b)$, there follows the finiteness of $\int(1+x^2)G(b,dx)$. Therefore, for all $\lambda\in\mathbb{R}$ we have $f''(\lambda)=-\int e^{i\lambda x}(1+x^2)G(b,dx)$. We denote $K(x)=\int_{-\infty}^{x}(1+y^2)G(b,dy)$. Then the function $g(\lambda)=\int e^{i\lambda x}K(dx)$ has all the derivatives at zero. Consequently [4], for all $n\in\mathbb{N}$ we have $\int|x|^nK(dx)<\infty$, whence we obtain at once that $\int|x|^nG(b,dx)<\infty$, i.e., condition 2) holds. Now we prove 1).

By assumption, for some nondecreasing and continuous function $F$ on $[a,b]$ we have $|E(\xi(s)-\xi(t))|\leqslant F(s)-F(t)$, $t<s$. But $E(\xi(s)-\xi(t))=\varkappa_1(t,s)=\gamma(s)-\gamma(t)+G^*(s)-G^*(t)$, where $G^*(t)=\int x\,G(t,dx)$. Obviously, $|G^*(s)-G^*(t)|\leqslant G_1(s)-G_1(t)$ for $t<s$, and, therefore, $|\gamma(s)-\gamma(t)|\leqslant|E(\xi(s)-\xi(t))|+|G^*(s)-G^*(t)|\leqslant(F(s)+G_1(s))-F(t)+G_1(t)),t<s$. Consequently, $\operatorname{Var}\gamma\big|_a^b\leqslant G_1(b)+F(b)-F(a)<\infty$.

## 3. Variations and the Relation with Multiple Integrals

Let $\mu$ be a process with independent increments on a measurable space $(\mathfrak{X},\mathfrak{A})$. We denote $\mathcal{P}_n=\{A_1\times\ldots\times A_n\colon A_i\in\mathfrak{A},\,A_i\cap A_j=\varnothing\,(i\neq j)\}$, $\mu^n(A_1\times\ldots\times A_n)=\mu A_1\cdot\ldots\cdot\mu A_n$. Thus, we have a new process $\mu^n$ (i.e., a random additive measure on the semiring $\mathcal{P}_n$ of sets) which, however, is not a process with independent increments. Nevertheless, one has

THEOREM 4. If the process $\mu$ is strongly continuous, then the process $\mu^n$ has a unique strongly continuous extension to the $\sigma$-algebra $\mathfrak{A}^n$. Moreover, if $m\in Z(\mu)$, then $m^n\in Z(\mu^n)$.

Proof. Without loss of generality, we can assume that $m\mathfrak{X}=1$, where $m$ is some fixed measure from $Z(\mu)$, i.e., $m$ is finite, continuous and for some sequence $\gamma_k\geqslant 0$ we have

$|E(\mu A)^k| \leqslant \gamma_k \cdot mA$ for each measurable $A$. We denote by $T$ the family of all partitions $\tau = \{A_1, \ldots, A_d\}$ of the set $\mathfrak{X}$, for which $mA_i = \frac{1}{d}$, $d \in \mathbb{N}$. If $\tau \in T$, then by $\mathcal{P}_n(\tau)$ we shall mean the family of all those sets from $\mathcal{P}_n$, which are products of sets from $\tau$. We denote by $A_n(\tau)$ the ring of sets generated by $\mathcal{P}_n(\tau)$. We note that by virtue of the continuity of $m$, for each $C \in \mathfrak{A}^n$ there exists a sequence $\tau_\ell \in T$ and $C_\ell \in A_n(\tau_\ell)$ such that $m^n(C \Delta C_\ell) \xrightarrow[\ell \to \infty]{} 0$. Therefore, it is sufficient to find a sequence $\alpha_k \geqslant 0$ such that $|E(\mu^n C)^k| \leqslant \alpha_k \cdot m^n C$ for all $C \in A_n(\tau)$, $\tau \in T$.

Let $D = \{1, \ldots, d\}$, let $D_n$ be the family of subsets of $D$ of order $n$, let $S$ be an arbitrary subset of $D_n$ of order $N$. Let $n \leqslant p \leqslant k \cdot n \leqslant d$. We show that the order of the set $\mathcal{E}^p = \{(S_1, \ldots, S_k) : S_i \in S, |\bigcup_1^k S_i| = p\}$ is not larger than $\delta(k,n,p) \cdot N \cdot d^{p-n}$, where $\delta(k,n,p)$ depends only on $k, n, p$.

We denote $S_1' = S_1$, $S_2' = S_2 \setminus S_1, \ldots, S_k' = S_k \setminus \bigcup_{i<k} S_i$; $z_i = |S_i'|$, $\mathcal{E}^p_{z_1,\ldots,z_k} = \{(S_1,\ldots,S_k) \in \mathcal{E}^p : |S_i'| = z_i\}$. We estimate the order $\mathcal{E}^p_{z_1,\ldots,z_k}$, where $z_i$ is a sequence of length $k$ such that $\sum z_i = p$, $z_1 = n$. The set $S_1$ can be chosen in at most $N$ ways since $S_1 \in S$, $S_2 = S_2' \cup (S_1 \cap S_2)$. The set $S_2'$ can be chosen in at most $C^{z_2}_{d-z_1} \leqslant d^{z_2}$ ways, and the set $S_1 \cap S_2$ in $C_n^{n-z_2}$ ways; consequently, $S_2$ can be chosen in at most $C^{n-z_2} \cdot d^{z_2}$ ways. Similarly, $S_3 = S_3' \cup (S_3 \cap (S_1 \cup S_2))$; $S_3'$ can be chosen in at most $d^{z_3}$ ways and $S_3 \cap (S_1 \cup S_2)$ in at most $C^{n-z_3}_{2n-z_2}$ ways; consequently, $S_3$ can be chosen in at most $C^{n-z_3}_{2n-z_2} d^{z_3}$ ways. Thus, $|\mathcal{E}^p_{z_1\ldots z_k}| \leqslant \delta_{z_1\ldots z_k} d^{z_2+\ldots+z_k} N = N \delta_{z_1\ldots z_k} d^{p-n}$, where $\delta_{z_1\ldots z_k} = C_n^{n-z_2} \cdots C^{n-z_k}_{(k-1)n-z_2-\ldots-z_{k-1}}$; consequently, $|\mathcal{E}^p| \leqslant N d^{p-n} \sum \delta_{z_1\ldots z_k} = \delta(k,n,p) N \cdot d^{p-n}$. The summation is taken over all $z_1, \ldots, z_k$ such that $\sum z_i = p$, $z_1 = n$, and, therefore, $\delta(k,n,p)$ depends only on $k, n, p$.

Assume now that $C = \bigcup_{j=1}^N C_j$, $C_j \in \mathcal{P}_n(\tau)$, $\tau = \{A_1, \ldots, A_d\}$. Since the partition can be always refined, we shall assume that $d \geqslant kn$. By assumption, each set $C_j$ has the form $A_{i_1} \times \ldots \times A_{i_n}$, where $i_\alpha \neq i_\beta$ $(\alpha \neq \beta)$. We associate to it the set $S(C_j) = \{i_1 \ldots i_n\} \in D_n$ and we denote $S^p = \{(j_1, \ldots j_k) : (S(C_{j_1}), \ldots, S(C_{j_k})) \in \mathcal{E}^p\}$, where for $S$ we have taken the set of all $S(C_j)$, $j = 1, \ldots, N$. Since for any permutation $\pi$ of the elements $\{1, \ldots, n\}$ we have $S(C_j) = S(C_j^\pi)$, where $C_j^\pi = A_{i_{\pi(1)}} \times \ldots \times A_{i_{\pi(n)}}$, it follows, obviously, that

$$|S^p| \leqslant n!^k \cdot |\mathcal{E}^p| \leqslant \delta'(k,n,p) \cdot N \cdot d^{p-n}, \text{ where } \delta'(k,n,p) = n!^k \delta(k,n,p).$$

Now we note that if $(S(C_{j_1}), \ldots, S(C_{j_k})) \in \mathcal{E}^p$, then $|E \mu^n C_{j_1} \cdots \mu^n C_{j_k}| \leqslant \gamma'(k,n) \cdot d^{-p}$, where $\gamma'(k,n)$ depends only on $k, n$ and $\gamma_1, \ldots, \gamma_k$. Consequently,

$$|E(\mu^n C)^k| \leqslant \sum_{j_1 \ldots j_k} |E \mu^n C_{j_1} \cdots \mu^n C_{j_k}| \leqslant \sum_{p=n}^{kn} \sum_{S^p} \leqslant \sum_{p=n}^{kn} |S^p| \cdot \gamma'(k,n) d^{-p} \leqslant \sum_{p=n}^{kn} \gamma'(k,n) \delta'(k,n,p) N \cdot d^{p-n} d^{-p} = \left(\sum_{p=n}^{kn} \gamma'(k,n) \delta'(k,n,p)\right) \cdot N \cdot d^{-n} = \alpha_k \cdot m^n C.$$

<u>Remark.</u> For the process $\mu^n$ one constructs the so-called multiple integral $I_n$ such that $I_n(1_C) = \mu^n C$, $C \in \mathfrak{A}^n$. Therefore, any statement regarding the measure $\mu^n$ can be considered as a statement on the multiple integral.

We consider now polynomials of $n$ variables with integer coefficients:

$$P_n(x_1, \ldots x_n) = \sum \frac{n! (-1)^{i_1 + \ldots + i_n}}{1^{i_1} 2^{i_2} \cdots n^{i_n} \cdot i_1! \cdots i_n!} x_1^{i_1} \cdots x_n^{i_n},$$

where the summation is over all nonnegative integers $i_1, \ldots i_n$ such that $1 \cdot i_1 + 2 i_2 + \ldots + n i_n = n$. These polynomials possess the following characteristic property: $\forall a_1, \ldots, a_N \in \mathbb{R}$, $N \geqslant n$

$$\sum_{\substack{i_\alpha \neq i_\beta \\ (\alpha\neq\beta)}} a_{i_1}\cdots a_{i_n} = P_n\Big(\sum_{i=1}^N a_i, \sum_{i=1}^N a_i^2, \dots, \sum_{i=1}^N a_i^n\Big).$$

We note that

a) $P_n(x,1,0,\dots,0)=H_n(x)$ is the Hermite polynomial of degree $n$,
$P_n(x,\sigma,0,\dots,0)=H_n(x,\sigma)$;

b) $P_n(x,x+t,\dots,x+t)=G_n(t,x)$ is the Poisson—Charlier polynomial of degree $n$,
$P_n(x,x,\dots,x)=x(x-1)\dots(x-n+1)$.

THEOREM 5. Let $\mu$ be a strongly continuous process with independent increments on a measurable space $(\mathfrak{X},\mathfrak{A})$. Then

1) $\forall A\in\mathfrak{A}$ in all $L^p(\Omega,\mathcal{F},P)$, $p<\infty$ there exists the limit $\mu_n A=\lim\sum\mu A_i^n$, where $\{A_1,\dots,A_N\}$ is a partition of $A$, $\max mA_i\to 0$, $m\in Z(\mu)$ (This limit is with respect to the filter $\mathcal{F}(\mu|_A)$, independent of $m$ according to Theorem 1.)

2) $\mu_n$ is a strongly continuous process with independent increments; moreover $Z(\mu)\subset Z(\mu_n)$.

3) $\mu^n A^n = P_n(\mu_1 A,\dots,\mu_n A)$ for each measurable $A$.

We carry out the proof of 1) and 3) by induction. Both statements are obvious for $n=1$ since $\mu_1=\mu$. Let $n>1, \tau=\{A_1,\dots,A_N\}$ be a partition of $A, N\geqslant n$. We denote $S_n(\tau)=\bigcup_{i_\alpha\neq i_\beta} A_{i_1}\times\dots\times A_{i_n}$, $\operatorname{rank}\tau=\max mA_i$. Obviously, $S_n(\tau)\subset A^n$ and $m^n(S_n(\tau)\,\Delta\, A^n)\to 0$ as $\operatorname{rank}\tau\to 0$. Consequently, by Theorem 4, if $m\in Z(\mu)$, then $\mu^n S_n(\tau)\to\mu^n A^n$ as $\operatorname{rank}\tau\to 0$ in all $L^p$, $p<\infty$.

From the formula for $P_n$ it is clear that $P_n(x_1,\dots,x_n)=Q_n(x_1,\dots,x_{n-1})+(-1)^n(n-1)!\,x_n$. If for $x_i$ one takes $\sum_{1\leqslant j\leqslant N}\mu A_j^i$, then by virtue of the mentioned property of $P_n$, we shall have

$$\mu^n S_n(\tau)=P_n(x_1,\dots,x_n)=Q_n(x_1,\dots,x_{n-1})+(-1)^n(n-1)!\,x_n.$$

By the induction hypothesis, for $i<n$ we have $x_i\to\mu_i A$ in all $L^p$, $p<\infty$, in the same place, consequently, also there we have $Q_n(x_1,\dots,x_{n-1})\to Q_{n-1}(\mu_1 A,\dots,\mu_{n-1}A)$, and thus, in all $L^p, p<\infty$ there exists the limit $\mu_n A=\lim x_n$ as $\operatorname{rank}\tau\to 0$; moreover, $\mu^n A^n=P_n(\mu_1 A,\dots,\mu_n A)$. The statement 2) is obvious.

Definition. The process $\mu_n$ is called the variation of $\mu$ of order $n$, and the measure $m_n=E\mu_n$ is its variational moment of order $n$.

Remark. In [1] one can find another proof of formula 3) for Wiener and Poisson processes.

## 4. Meaning of the Variational Moments

We define the generating function for the sequence of variational moments:

$$F_\mu(z)=\sum_{n=1}^\infty m_n\frac{z^n}{n!}.$$

We assume that this series converges absolutely in some neighborhood of zero $|z|<R$, where $0<R\leqslant\infty$. Then, we have

THEOREM 6. $Ee^{it\mu}=e^{F_\mu(it)}$, $|t|<R$, and, consequently, $m_n A$ is the cumulant of order $n$ of the random variable $\mu A$.

Proof. Representing $\mu A$ as $\sum_{i=1}^{N}\mu A_i$ and letting the rank of the partition go to zero, we can obtain without difficulty that $E(\mu A)^n=\sum\frac{n!}{\alpha_1!\ldots\alpha_s!\,s!}m_{\alpha_1}A\cdot\ldots\cdot m_{\alpha_s}A$, where the summation is over all integers $\alpha_1,\ldots\alpha_s>0$ such that $\alpha_1+\ldots+\alpha_s=n$, $1\leqslant s\leqslant n$.

But then for $|z|<R$ we have

$$Ee^{z\mu}=1+\sum\frac{1}{s!}m_{\alpha_1}\cdot\ldots\cdot m_{\alpha_s}\frac{z^{\alpha_1}}{\alpha_1!}\cdot\ldots\cdot\frac{z^{\alpha_s}}{\alpha_s!}=1+\sum_{s=1}^{\infty}\frac{1}{s!}F_\mu(z)^s=e^{F_\mu(z)}.$$

## 5. Characterization of Wiener and Poisson Processes in Terms of Variations

A process $\mu$ with independent increments on a measurable space $(\mathfrak{X},\mathfrak{A})$ is said to be a Wiener process if each random variable $\mu A$, where $A\in\mathfrak{A}$, has a normal distribution. In this case the strong continuity of $\mu$ is equivalent to the fact that $E\mu$ and $D\mu$ are continuous measures on $(\mathfrak{X},\mathfrak{A})$.

THEOREM 7. For a strongly continuous process $\mu$ with independent increments the following statements are equivalent:

1. $\mu$ is a Wiener process;
2. $\forall n>2$ $\mu_n=0$;
3. $\forall n>2$ $m_n=0$;
4. $\exists n\geqslant 2$ $m_{2n}=0$.

Proof. The implications $1.\Rightarrow 2.\Rightarrow 3.\Rightarrow 4$ are obvious. From statement 3 we obtain that $\mu_2=m_2$ and by Theorem 6 we obtain statement 1. It remains to show that $4.\Rightarrow 3.$ From the definition of the variation it is clear that $\forall i,j\geqslant 1$ $E(\mu_i-m_i)(\mu_j-m_j)=m_{i+j}\Rightarrow\forall a,b\in\mathbb{R}$ $0\leqslant D(a(\mu_i-m_i)+b(\mu_j-m_j))=a^2m_{2i}+2abm_{i+j}+b^2m_{2j}\Rightarrow\forall i,j\geqslant 1$ $m^2_{i+j}\leqslant m_{2i}\cdot m_{2j}$. Therefore $m_{2n}=0$ implies $m_{n+i}=0$ $\forall i\geqslant 1$. Let $m_{2n_0}=0$. In the natural segment $[n_0+1,2n_0]$ we find a least even number $2n_1$ and we have again $m_{n_1+i}=0$ $\forall i\geqslant 1$. In a similar manner we construct a sequence of natural numbers $n_k$ for which $2n_k$ is the smallest even number in $[n_{k-1}+1,2n_{k-1}]$. In this case $m_{n_k+i}=0$ $\forall i\geqslant 1$. Obviously, for some $k$ we have $n_k=4\Rightarrow\forall i\geqslant 5$ $m_i=0$. Since $m_6=m_{2\cdot 3}=0$, we have $m_4=m_{3+1}=0$. But $4=2\cdot 2$, and, consequently, $m_3=m_{2+1}=0$.

COROLLARY. Let $\xi(t),a\leqslant t\leqslant b$, be a stochastic continuous process with independent increments such that $\xi(a)=0$, $E\xi(t)=0$. Then $\xi(t)$ is a Wiener process $\Longleftrightarrow\exists n\geqslant 2$ $E|\xi(s)-\xi(t)|^{2n}=0(s-t)$.

A process with independent increments on a measurable space $(\mathfrak{X},\mathfrak{A})$ will be called a Poisson process if each random variable $\mu A$, where $A\in\mathfrak{A}$, has a Poisson distribution with parameter $m_1A=E\mu A$. In this case the condition of strong continuity is equivalent to the fact that $m_1$ is a continuous measure on $(\mathfrak{X},\mathfrak{A})$.

THEOREM 8. For a strongly continuous process $\mu$ with independent increments, the following statements are equivalent:

1. $\mu$ is a Poisson process;
2. $\forall n$ $\mu_n=\mu$;
3. $\forall n$ $m_n=m_1$.

Proof. The implications I.$\Rightarrow$3. and 2.$\Rightarrow$3. are obvious, while 3.$\Rightarrow$I. follows from Theorem 6. We prove that 3.$\Rightarrow$2. As mentioned before, $E(\mu_i - m_i)(\mu_j - m_j) = m_{i+j}$. Consequently, $E(\mu_i - \mu_j)^2 = 0$ if all $m_i = m_1$.

## 6. Applications to Locally Weakly Dependent Processes

Definitions. 1) We shall say that the processes $\mu, \nu$ have joint independent increments if $\forall A_1, \ldots, A_N \in \mathfrak{A}$, $A_i \cap A_j = \phi\,(i \neq j)$ $(\mu A_1, \nu A_1), \ldots, (\mu A_N, \nu A_N)$ are independent random vectors.

2) The strongly continuous processes $\mu, \nu$ with independent increments will be said to be locally weakly dependent (LWD) if for some measure $m \in Z(\mu) \cap Z(\nu)$ and any $\alpha, \beta \in \mathbb{N}$ one has $E(\mu A)^\alpha (\nu A)^\beta = O(mA)$ as $mA \to 0$.

We give without proof a statement which can be easily obtained with the aid of Theorem 6.

THEOREM 9. Let $\mu, \nu$ be strongly continuous processes having joint independent increments. Let $F_\mu$ and $F_\nu$ be entire functions. Then $\mu, \nu$ are independent processes $\Leftrightarrow \mu, \nu$ are LWD.

COROLLARIES. Let $\mu, \nu$ be strongly continuous processes having joint independent increments.

If $\mu, \nu$ are Poisson processes, then $\mu + \nu$ is a Poisson process $\Leftrightarrow \mu, \nu$ are independent processes.

2. If $\mu$ is a Wiener process and $F_\nu$ is an entire function, then $\mu, \nu$ are independent $\Leftrightarrow \exists m \in Z(\mu) \cap Z(\nu)\ \forall \alpha \in \mathbb{N}\ E\mu A \nu A^\alpha = O(mA)$.

## 7. Stochastic Integrals

Here we prove that the polynomials $P_n$, introduced in Sec. 3, play the same role in the Itô integration as the standard polynomials $x^n$ in the Riemann integral.

Let $\xi(t)$, $0 \leq t \leq a$, be a strongly continuous process with independent increments such that $\xi(a) = 0$. We denote by $\mu$ the strongly continuous extension of the process $\xi(t)$ to the $\sigma$-algebra of Borel subsets of $[0, a]$, which exists according to Theorem 2.

It is known (see [1]) that there exists a unique continuous linear mapping $I_n : L^2([0,a]^n, m^n) \longrightarrow L^2(P)$, where $m \in Z(\mu)$ is such that $I_n(1_C) = \mu^n C$ for all Borel sets $C \subset [0,a]^n$. The operator $I_n$ is called a multiple integral and is usually denoted by $I_n(\varphi) = \int \varphi(x)\, d\mu^n(x)$. We denote

$$C_n(t) = \{(t_1, \ldots, t_n) : 0 \leq t_1 \leq \ldots \leq t_n \leq t\}, \text{ where } 0 \leq t \leq a.$$

THEOREM 10. Let $\varphi \in L^2(C_{n+1}(a), m^{n+1})$, $f(t) = \int\limits_{C_n(t)} \varphi(t_1, \ldots, t_n, t)\, d\mu^n(t_1, \ldots, t_n)$. Then

1) $f$ is a progressively measurable function;

2) $\int_0^t f(s)\, d\xi(s) = \int\limits_{C_{n+1}(t)} \varphi(t_1, \ldots, t_{n+1})\, d\mu^{n+1}(t_1, \ldots, t_{n+1})$.

The proof is obvious for step functions and they are dense in $L^2$.

COROLLARY. We denote by $\xi_n(t) = \mu_n[0,t)$ the variations of order $n$. Then

$$\int_0^t P_n(\xi_1(s), \ldots, \xi_n(s))\, d\xi(s) = \frac{1}{n+1} P_{n+1}(\xi_1(t), \ldots, \xi_{n+1}(t)) \text{ for } 0 \leq t \leq a.$$

We consider two special cases.

1. Let $\xi(t)$ be the standard Wiener process, i.e., $E\xi(t)=0, D\xi(t)=t$. Then, as already known, $\xi_n(t)=0$ for $n>2, \xi_2(t)=t$. Consequently, we have (see also [1] or [2])

$$\int_0^t H_n(\xi(s),s)\,d\xi(s)=\frac{1}{n+1}H_{n+1}(\xi(t),t).$$

2. Let $\xi(t)$ be the standard Poisson process, i.e., $E\xi(t)=t$. By Theorem 8, we have $\xi_n(t)=\xi(t)$. Consequently,

$$\int_0^t \xi(s)\cdot(\xi(s)-1)\cdot\ldots\cdot(\xi(s)-n+1)\,d\xi(s)=\frac{1}{n+1}\xi(t)(\xi(t)-1)\cdot\ldots\cdot(\xi(t)-n).$$

From this formula it is clear that $\int_0^t \xi(s)^n d\xi(s)=q_{n+1}(\xi(t))$, where the polynomials $q_n$ can be found recurrently. One can also show that

$$\int_0^t e^{q\xi(s)}d\xi(s)=\frac{e^{q\xi(t)}-1}{e^q-1}, \text{ where } q\in\mathbb{C},\ e^q\neq 1.$$

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