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ON LOGARITHMICALLY CONCAVE MEASURES AND THEIR APPLICATIONS TO RANDOM PROCESSES LINEARLY GENERATED BY INDEPENDENT VALUES

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C. Borell introduced [1] several definitions of s-convexity for measures on a vector space. Here is one of them.

Let (E,μ) be a vector space with measure (not necessary finite). A measure μ is called log-convex (or concave) if for all nonempty μ-measurable sets $A, B \subset E$ and for all $t, s \geqslant 0$, $t+s=1$

$$\mu(tA+sB) \geqslant \mu(A)^t \mu(B)^s, \tag{1}$$

(the set $tA+sB$ is supposed to be measurable). Some sufficient conditions for a measure μ to be log-concave are established in this paper. Unfortunately, the author is not familiar with [2], where this topic is probably under discussion. The main assertion is as follows.

Theorem. *The measure μ on $\mathbb{R}^n$ with density*

$$\exp\{-|x_1|^p - \cdots - |x_n|^p\}$$

is logarithmically concave by $p \geqslant 1$.

We note that the log-concavity property does not depend on the way of norming and is stable with respect to linear maps and to restrictions to convex subsets. The Lebesgue measure is log-concave (a consequence of the Brunn–Minkowski inequality). For the n-dimensional uniform distribution λ_N on the convex set

$$\{\, x \in \mathbb{R}^N \colon |x_1|^p + \cdots + |x_N|^p \leqslant N \,\}$$

the sequence of n-dimensional projections $\lambda_N \pi^{-1}$ on $\mathbb{R}^n$ converges in variation to a measure on $\mathbb{R}^n$ with density

$$C \exp(|x_1|^p + \cdots + |x_N|^p),$$

where C is a normalizing constant.

In fact, what was just stated is a sketch of a proof. For $p=2$, we obtain the well-known concavity of Gaussian measures.

We note one more consequence of the theorem. Let $\{\, e_n : n \geqslant 1 \,\}$ be a sequence of independent identically distributed random variables such that its common distribution is concentrated on an interval in $\mathbb{R}$ and has the density $C\exp(-|x|^p)$, $p \geqslant 1$. Consider a bounded random process representable in the form

$$\xi(t) = \sum_{n=1}^{\infty} a_n(t) e_n$$

and the distribution function connected with it:

$$F(x) = \mathbf{P}\{\sup_t \xi(t) \leqslant x\}.$$

Corollary. *The function $\log F(x)$ is concave on the interval $(a, +\infty)$, where*

$$a = \inf\{\, x \in \mathbb{R} : F(x) > 0 \,\},$$

and, hence, the distribution F has a density on $(a, +\infty)$ and can have a jump at the point a ($a > -\infty$).

In the Gaussian case the assertion about the density and the jump is the well-known Tsirel'son theorem.

Literature Cited

1. C. Borell, "Convex measures on locally convex spaces," *Ark. Mat.*, **12**, No. 2, 239–252 (1974).
2. B. G. Hansen, "On log-concave and log-convex infinitely divisible sequences and densities," *Ann. Probab.*, **16**, No. 4, 1832–1839 (1988).

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