Journal of Mathematical Sciences, Vol. 88, No. 1, 1998

ISOPERIMETRIC PROBLEM ON THE REAL LINE

S. G. Bobkov UDC 519.2

The isoperimetric problem on the real line for distributions with continuous positive densities is considered. Necessary and sufficient conditions under which the intervals $(-\infty, a)$ are extremal are suggested. Bibliography: 5 titles.

Denote by $\mathfrak{F}$ the family of all probability distributions F on the real line $\mathbb{R}$, concentrated on an open (finite or not) interval $(a_F, b_F) \subseteq \mathbb{R}$, that have a continuous positive density f on this interval. In this paper, necessary and sufficient conditions are established for a distribution F from $\mathfrak{F}$ to have the following property: for all Borel subsets $A \subseteq \mathbb{R}$, $F(A) = p > 0$,

$$F(A^h) \geq F(F^{-1}(p) + h). \tag{1}$$

Here A^h stands for the open h-neighborhood $\{x \in \mathbb{R} : \exists\, a \in A, |x - a| < h\} = A + (-h, h)$ of A and F^{-1} denotes the inverse function $(0,1) \to (a_F, b_F)$ for the distribution function $F(x) = F((-\infty, x))$ restricted to (a_F, b_F) (where F strictly increases).

For intervals $A = (-\infty, F^{-1}(p))$ inequality (1) turns into an equality, hence (1) expresses the fact that intervals are extremal in the isoperimetric problem for the measure F, i.e., they minimize the value of $F(A^h)$ under the condition $F(A) = p$. This property can be reformulated in the following equivalent way. If a random variable ξ has the distribution F, then for every Lipshitz function $g : \mathbb{R} \to \mathbb{R}$ with Lipshitz constant ≤ 1 the random variable $\eta = g(\xi)$ deviates from its quantiles $m_p(\eta)$ not more than ξ deviates from $m_p(\xi)$: for all $p \in (0,1)$ and $h > 0$

$$\mathbf{P}\{\eta - m_p(\eta) \geq h\} \leq \mathbf{P}\{\xi - m_p(\xi) \geq h\};$$
$$\mathbf{P}\{\eta - m_p(\eta) \leq -h\} \leq \mathbf{P}\{\xi - m_p(\xi) \leq -h\}.$$

Inequality (1) is well known for Gaussian measure. It was established by V. N. Sudakov and B. Tsirel′son [1] and C. Borell [2] in a much more general situation, in particular, for standard multivariate (or even infinite-variate) Gaussian measures and for A^h properly defined. The proof of this deep assertion is based on the Schmidt theorem on the isoperimetric property of balls on the surface of a sphere, and there was no need to investigate the one-dimensional case separately. M. Talagrand [3] obtained an inequality of isoperimetric type for the product-measure E^n constructed by the two-sided exponential distribution E. In particular, analysis of the univariate case showed that E satisfies relation (1). It was shown in [4] how the infinite-dimensional problem for a product measure and the uniform distance can be solved, provided the one-dimensional problem has been solved. Afterward, in [5] it was established that a logarithmically concave distribution F on the real line satisfies relation (1) if and only if it is symmetric (with respect to its median). The role of logarithmic concavity (which means the concavity of $\log f$ on (a_F, b_F)) can be explained by the following characterization. Consider a function I_F of the variable $p : 0 < p < 1$ $I_F(p) = f(F^{-1}(p))$.

Theorem. *A distribution $F \in \mathfrak{F}$ has property (1) if and only if*

(i) *F is symmetric with respect to its median, or, in terms of I_F, for any $0 < p < 1$*

$$I_F(1-p) = I_F(p);$$

(ii) *For all p, q, $0 < p, q < 1$ such that $p + q < 1$,*

$$I_F(p+q) \leq I_F(p) + I_F(q).$$

Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 216, 1994, pp. 5–9. Original article submitted February 4, 1994.

1072-3374/98/8801-0003$20.00

It can be seen that $F \to I_F$ specifies a one-to-one (up to a shift) correspondence between $\mathfrak{F}$ and the family of all continuous positive functions on $(0,1)$. On the other hand, the class of all symmetric logarithmically concave distributions is not very much less than the class of distributions satisfying (i) and (ii), as the following assertion shows.

Corollary. *Let $F \in \mathfrak{F}$ satisfy* (1), *the density f being differentiable on (a_F, b_F). Then F has exponential moments and, moreover, the increasing map $T : \mathbb{R} \to (a_F, b_F)$ transferring a two-sided exponential distribution E into F (i.e., such that $ET^{-1} = F$) is Lipshitzian.*

Proof of the theorem. Necessity. Rewrite relation (1) for the interval $A = (b, +\infty)$, $b = F^{-1}(1-p)$ with F-measure equal to $p \in (0,1)$:

$$1 - F\big(F^{-1}(1-p) - h\big) \geq F\big(F^{-1}(p) + h\big). \tag{2}$$

The Taylor expansion of both sides of (2) at $h = 0$ implies $I_F(1-p) \geq I_F(p)$. Substituting $1-p$ for p, we come to the inverse inequality, hence (i) holds. The same Taylor expansion for the interval $A = (a,b) \subseteq (a_F, b_F)$ of F-measure p gives $f(a) + f(b) \geq I_F(p)$. Let $F(a) = q$; then $F(b) = p + q$, and the last inequality obtains the form

$$I_F(q) + I_F(p+q) \geq I_F(p). \tag{3}$$

Now if we substitute p' for $1-(p+q)$, q' for q, and use (i), then (3) coincides with (ii) (i.e., (3) is equivalent to (ii) under condition (i)). Hence (ii) is also proved.

Sufficiency. First we note that because of the nonatomicity of F it is sufficient to prove inequality (1) only for open A such that they can be represented as the union of a finite number of open intervals (possibly, infinite) of positive F-measure. Denote by $\mathfrak{L}$ the class of all such sets. Evidently, if $A \in \mathfrak{L}$, then $A^h \in \mathfrak{L}$ for all $h > 0$. On the class $\mathfrak{L}$ we consider a "surface measure" F^+, which is by definition specified as

$$F^+(A) = \lim_{h \to 0+} \big(F(A^h) - F(A)\big)/h. \tag{4}$$

This measure F^+ has the following two properties.

(1) If the closures of intervals Δ_i, $1 \leq i \leq n$, composing $A \in \mathfrak{L}$ are pairwise disjoint, then

$$F^+(A) = \sum_i F^+(\Delta_i). \tag{5}$$

(2) For intervals of the form $\Delta = (a,b)$

$$F^+(\Delta) = f(a) + f(b), \tag{6}$$

and $f = 0$ on $(-\infty, a_F) \cup (b_F, +\infty)$.

Properties (1)–(2) and hypothesis (i)–(ii) imply that for all $A \in \mathfrak{L}$ such that $p = F(A) < 1$ we have

$$F^+(A) \geq I_F(p). \tag{7}$$

Indeed, for intervals $A = (a,b)$ inequality (7) turns into (3) due to (6). As we saw, inequality (3) is equivalent to (ii) if (i) holds.

Now consider the general case. Let $A \in \mathfrak{L}$, and let $A' \subset \mathbb{R}$ be constructed by additing to A the end points of intervals composing A such that they touch other intervals. Then we have $F(A') = F(A) = p$, $A' \in \mathfrak{L}$, and the new intervals Δ_i, $1 \leq i \leq n$ composing A' are separated from each other. Since (5) is valid for A', we get $F^+(A') \leq F^+(A)$. Applying inequality (7) and inequality (ii) with a finite number of summands to the new intervals Δ_i (we put $p_i = F(\Delta_i)$), we have

$$F^+(A) \geq F^+(A') = \sum_i F^+(\Delta_i) \geq \sum_i I_F(p_i) \geq I_F(p).$$

Hence, (7) is proved. It remains only to understand (1) as an integral version of (7). Let us fix a parameter $\sigma > 1$ and consider the family of continuous nondecreasing functions $R_h : (0,1] \to (0,1]$ specified by the equalities

$$R_h(p) = F\big(F^{-1}(p) + h/\sigma\big), \qquad 0 < p < 1, \tag{8}$$
$$R_h(1) = 1.$$

It is sufficient to prove that for all $A \in \mathfrak{L}$ with $F(A) = p$ the inequality

$$F(A^h) \geq R_h(p) \tag{9}$$

holds. Then turning $\sigma \to 1$ and using the continuity of F, we get (1). Inequality (9) is evident if $F(A) = 1$, hence we may restrict ourselves to consideration of only the sets $A \in \mathfrak{L}$ of measure $F(A) = p \in (0,1)$. Fixing such a set, we put

$$J(A) = \big\{ h > 0 : \ (9) \text{ is valid for all } \ h' \in (0,h] \big\}.$$

Evidently, if $h_n < h$, $h_n \to h$, $h_n \in J$, then $h \in J$, too. This is why the relation $J = (0,\infty)$ follows from the properties below:

(a) $\varepsilon \in J$ for all $\varepsilon > 0$ sufficiently small;

(b) if $h \in J$, then $h + \varepsilon \in J$ for all $\varepsilon > 0$ sufficiently small.

For small $\varepsilon > 0$ consider the Taylor expansion for $R_\varepsilon(p)$ at the point $\varepsilon = 0$ coming from (8), and the corresponding expansion for $F(A^\varepsilon)$ coming from (4):

$$R_\varepsilon(p) = p + I_F(p)\, \varepsilon/\sigma + o(\varepsilon),$$
$$F(A^\varepsilon) = p + F^+(A)\, \varepsilon + o(\varepsilon).$$

Comparing these two expansions and taking into account (7), we find that $F(A^\varepsilon) > R_\varepsilon(p)$ for all $\varepsilon > 0$ sufficiently small and, hence, (a) proves to be true.

Now we prove (b). Let $h \in J(A)$, hence, $F(A^h) \geq R_h(p)$. If $F(A^h) = 1$, then $F(A^{h+\varepsilon}) = 1$, i.e., there is nothing to prove. If $F(A^h) < 1$, we consider two cases. If $F(A^h) > R_h(p)$, then this inequality is conserve by continuity for all $h' = h + \varepsilon$ close to h, and again there is nothing to prove. If, finally, $F(A^h) = R_h(p)$, then one should use the following property of the family R_h, which can be immediately verified: for all $p \in (0,1]$ and $h, \varepsilon > 0$,

$$R_{h+\varepsilon}(p) = R_h\big(R_\varepsilon(p)\big). \tag{10}$$

Applying (a) to the set A^h, we get for $\varepsilon > 0$ sufficiently small: $\varepsilon \in J(A^h)$, i.e.,

$$F\big((A^h)^\varepsilon\big) \geq R_\varepsilon\big(F(A^h)\big) = R_\varepsilon\big(R_h(p)\big).$$

Taking into account (10) and the equality $A^{h+\varepsilon} = (A^h)^\varepsilon$, we come to the inequality $F(A^{h+\varepsilon}) \geq R_{h+\varepsilon}(p)$. This proves (b) and, hence, the theorem.

Proof of the corollary. The relation $I_E(p) = \min(p, 1-p)$ shows that the Lipschitz property of the function T means the existence of a constant $c > 0$ such that $I_F(p) \geq c\, I_E(p)$ for all $0 < p < 1$. By the theorem just proved I_F is symmetric with respect to $p = {}^1\!/_2$ and, since $F \in \mathfrak{F}$, I_F is separated from 0 in any interval of the form $(\varepsilon, 1-\varepsilon)$, $\varepsilon > 0$. It follows that the last inequality may be written in the form of the condition

$$d = \liminf_{p \to 0+} I_F(p)/p > 0.$$

If $I_F(0+) > 0$, then there is nothing to prove. Let $I_F(0+) = 0$, and suppose the converse, i.e., that $d = 0$. Since I_F is differentiable on $(0,1)$, we have $I_F(p+\varepsilon) = I_F(p) + I_F'(p)\varepsilon + o(\varepsilon)$ as $\varepsilon \to 0$, $0 < p < 1$. On the other hand, by the inequality (ii) (which also is a consequence of the statement of the theorem), $I_F(p+\varepsilon) \leq I_F(p) + I_F(\varepsilon)$ as $\varepsilon > 0$. Hence $I_F'(p) \leq I_F(\varepsilon)/\varepsilon + o(1)$ as $\varepsilon \to 0+$. On the strength of the assumption $d = 0$ we get $I'_F(p) \leq 0$, i.e., I_F does not increase. But I_F is symmetric, hence $I_F = \text{const}$. This contradiction completes the proof.

Translated by V. Sudakov.

REFERENCES

1. V. N. Sudakov and B. S. Tsirel'son, "Extremal properties of half-spaces for spherically invariant measures," *J. Sov. Math.*, **9**, 9–18 (1978).
2. C. Borell, "The Brunn–Minkowski inequality in Gauss space," *Invent. Math.*, **30**, 207–216 (1975).
3. M. Talagrand, "A new isoperimetric inequality and the concentration of measure phenomenon," *Lect. Notes Math.*, **1469**, 94–124 (1989–1990).
4. S. Bobkov, "Isoperimetric problem for uniform enlargement," Univ. of North Carolina at Chapel Hill, Dept. of Statistics. Tech. Report No. 394 (1993).
5. S. Bobkov, "Extremal properties of half-spaces for log-concave distributions," Univ. of North Carolina at Chapel Hill, Dept. of Statistics. Tech. Report No. 396 (1993).