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Poincaré's inequalities and Talagrand's concentration phenomenon for the exponential distribution

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Summary. We present a simple proof, based on modified logarithmic Sobolev inequalities, of Talagrand's concentration inequality for the exponential distribution. We actually observe that every measure satisfying a Poincaré inequality shares the same concentration phenomenon. We also discuss exponential integrability under Poincaré inequalities and its consequence to sharp diameter upper bounds on spectral gaps.

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1 Introduction

The concentration phenomenon for the canonical Gaussian measure γ_n on $\mathbb{R}^n$ (cf. e.g. [L-T], [Le2]) expresses that for every Borel set A in $\mathbb{R}^n$ with $\gamma_n(A) \geqq \frac{1}{2}$ and every $r \geqq 0$,

$$\gamma_n(A + rB_2) \geqq 1 - e^{-r^2/2}\,, \tag{1.1}$$

where B_2 is the Euclidean unit ball in $\mathbb{R}^n$. Equivalently, if f is a Lipschitz map on $\mathbb{R}^n$ with Lipschitz coefficient $\|f\|_{\mathrm{Lip}} \leqq 1$ (with respect to the Euclidean metric), for every $t \geqq 0$,

$$\gamma_n(f \geqq M + t) \leqq e^{-t^2/2}\,, \tag{1.2}$$

where M is either the mean or a median of f for γ_n. A few years ago, M. Talagrand [Ta1] proved an isoperimetric inequality for the product measure of the exponential distribution which implies the following concentration property. Let ν_n be the product measure on $\mathbb{R}^n$ when each factor is endowed

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with the measure of density $\frac{1}{2}e^{-|x|}$ with respect to Lebesgue measure. Then, for every Borel set A with $\nu_n(A) \geqq \frac{1}{2}$ and every $r \geqq 0$,

$$\nu_n(A + \sqrt{r}B_2 + rB_1) \geqq 1 - e^{-r/K} , \tag{1.3}$$

where $K > 0$ is some numerical constant and where B_1 is the ℓ^1 unit ball in $\mathbb{R}^n$, i.e.

$$B_1 = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n;\ \sum_{i=1}^{n} |x_i| \leqq 1 \right\} .$$

A striking feature of (1.3) is that it may be used to improve some aspects of the Gaussian concentration (1.1), especially for cubes (cf. [Ta1], [Ta2]).

As for (1.1), inequality (1.3) may be translated equivalently on functions in the following way (see the end of Sect. 2 for details). For every real-valued function f on $\mathbb{R}^n$ such that $\|f\|_{\mathrm{Lip}} \leqq \alpha$ and

$$|f(x) - f(y)| \leqq \beta \sum_{i=1}^{n} |x_i - y_i|, \quad x, y \in \mathbb{R}^n ,$$

for every $t \geqq 0$,

$$\nu_n(f \geqq M + t) \leqq \exp\left(-\frac{1}{K} \min\left(\frac{t}{\beta}, \frac{t^2}{\alpha^2} \right) \right) \tag{1.4}$$

for some numerical constant $K > 0$ where M is either the mean or a median of f for ν_n. By Rademacher's theorem, the hypotheses on f are equivalent to saying that f is almost everywhere differentiable with

$$\sum_{i=1}^{n} |\partial_i f|^2 \leqq \alpha^2 \quad \text{and} \quad \max_{1 \leqq i \leqq n} |\partial_i f| \leqq \beta \quad \text{a.e.}$$

Similar inequalities hold for products of the one-sided exponential distribution.

The first aim of this work is to present an elementary proof of inequality (1.4) (and thus (1.3)) based on logarithmic Sobolev inequalities. An alternate proof, however close to Talagrand's ideas, has already been given by B. Maurey using inf-convolution [Ma] (see also [Ta3]). M. Talagrand himself obtained recently another proof as a consequence of a stronger transportation cost inequality [Ta4]. Our approach is simpler even than the transportation method, and connects to a well known theory. To illustrate it, let us first recall that the deviation inequality (1.2) may be shown to follow from the logarithmic Sobolev inequality for the Gaussian measure γ_n ([Gr]) that expresses that, for every smooth function g on $\mathbb{R}^n$,

$$\int g^2 \log g^2 \, d\gamma_n - \int g^2 \, d\gamma_n \log \int g^2 \, d\gamma_n \leqq 2 \int |\nabla g|^2 \, d\gamma_n , \tag{1.5}$$

where $|\nabla g|$ denotes the Euclidean length of the gradient ∇g of g. The argument goes back to I. Herbst and to E.B. Davies and B. Simon [D-S] and has been revived recently in [A-M-S], [A-S], [Le1], [Le3]. It consists in applying (1.5) to $g^2 = e^{\lambda f}$ where $\lambda \in \mathbb{R}$ and $\|f\|_{\mathrm{Lip}} \leqq 1$ to deduce the differential inequality

$$\lambda F'(\lambda) - F(\lambda) \log F(\lambda) \leqq \frac{\lambda^2}{2} F(\lambda)$$

on the Laplace transform $F(\lambda) = \int e^{\lambda f} \, d\gamma_n$ of f. Integration of this inequality then easily yields

$$F(\lambda) \leqq e^{\lambda \int f d\gamma_n + \lambda^2/2}, \quad \lambda \in \mathbb{R},$$

from which (1.2) follows together with Chebyshev's inequality.

Following this procedure in case of the exponential distribution would require to determine the appropriate logarithmic Sobolev inequality satisfied by v_n. We cannot hope for an inequality such as (1.5) simply because the preceding argument would imply that linear functionals have a Gaussian tail for v_n. To investigate analogues of (1.5) for v_n, it is enough, by the fundamental product property of logarithmic Sobolev inequalities, to consider the dimension 1. We work here with the one-sided exponential distribution. One first inequality may be deduced from the Gaussian logarithmic Sobolev inequality. Given a smooth function f on $\mathbb{R}^+$, apply (1.5) in dimension 2 to $g(x, y) = f((x^2 + y^2)/2)$. Let $\tilde{v}_1$ denote the one-sided exponential distribution with density e^{-x} with respect to Lebesgue measure on $\mathbb{R}^+$, and let $\tilde{v}_n$ denote the product measure on $\mathbb{R}^{+n}$. Then

$$\int f^2 \log f^2 \, d\tilde{v}_1 - \int f^2 \, d\tilde{v}_1 \log \int f^2 \, d\tilde{v}_1 \leqq 4 \int x f'(x)^2 \, d\tilde{v}_1(x) .$$

By the product property of entropy, for every smooth f on $\mathbb{R}^{+n}$,

$$\int f^2 \log f^2 \, d\tilde{v}_n - \int f^2 \, d\tilde{v}_n \log \int f^2 \, d\tilde{v}_n \leqq 4 \int \sum_{i=1}^{n} x_i \partial_i f(x)^2 d\tilde{v}_n(x) . \qquad (1.6)$$

It does not seem however that this logarithmic Sobolev inequality (1.6) can yield the concentration property (1.4) (for $\tilde{v}_n$) via the preceding Laplace transform approach. In a sense, this negative observation is compatible with the fact that (1.4) improves upon some aspects of the Gaussian concentration which is a consequence of (1.5) as (1.6) is one! We thus have to look for some other version of the logarithmic Sobolev inequality for the exponential distribution. To this aim, let us observe that, at the level of Poincaré inequalities, there are two distinct inequalities. For simplicity, let us deal again only with $n = 1$. The first one, in the spirit of (1.6), indicates that

$$\int f^2 \, d\tilde{v}_1 - \left(\int f \, d\tilde{v}_1 \right)^2 \leqq \int x f'(x)^2 \, d\tilde{v}_1(x) .$$

This may be shown, either from the Gaussian Poincaré inequality as before, with however a worse constant, or by noting that the first eigenvalue of the Laguerre generator with invariant measure $\tilde{v}_1$ is 1 (cf. [K-S]. By the way, that 4 is the best constant in (1.6) is an easy consequence of the Laplace transform method described above. Namely, if (1.6) holds with a constant $C < 4$, a function f, on $\mathbb{R}^+$ for simplicity, such that $x f'(x)^2 \leqq 1$ almost everywhere would be such that $\int e^{f^2/4} \, d\tilde{v}_1 < \infty$. But the example of $f(x) = 2\sqrt{x}$ contradicts this consequence and we thus recover in this simple way the main result of [K-S].). The second inequality appeared in the work by M. Talagrand [Ta1], actually going back to [Kl], and states that

$$\int f^2 \, d\tilde{\nu}_1 - \left(\int f \, d\tilde{\nu}_1 \right)^2 \leqq 4 \int f'^2 \, d\tilde{\nu}_1 \, . \tag{1.7}$$

These two inequalities are not comparable and, in a sense, we are looking for an analogue of (1.7) for entropy. Our first main result in this direction will be that for any Lipschitz function f on $\mathbb{R}^+$ with $|f'| \leqq c < 1$ a.e.,

$$\int f e^f \, d\tilde{\nu}_1 - \int e^f \, d\tilde{\nu}_1 \log \int e^f \, d\tilde{\nu}_1 \leqq \frac{2}{1-c} \int f'^2 e^f \, d\tilde{\nu}_1 \, . \tag{1.8}$$

This inequality is the right inequality to perform, after tensorisation in n-dimension, the approach based on a differential inequality on Laplace transforms and to reach, in this way, Talagrand's concentration (1.4). This result is moreover a further step in the program of [Le3] that intends to investigate Talagrand's inequalities for product measures [Ta3] from a functional point of view based on logarithmic Sobolev inequalities.

It actually turns out that there is a general principle behind the example of the exponential distribution and that the Poincaré inequality (1.7) plays a crucial role in this question. As a main result, we will namely prove that any measure satisfying such a Poincaré inequality will satisfy an inequality on entropy such as (1.8), and therefore a concentration result similar to the one for the exponential measure. The conclusion will be sharp in the sense that, starting from the Poincaré inequality for the Gaussian measure, we will even recover Gross's logarithmic Sobolev with its best constant. The general result strengthens and clarifies some aspects of a prior result of [Ta3] (in the context of penalties) that deals with a stronger condition than Poincaré's inequality.

We prove (1.8) in Sect. 2, as well as the application to Talagrand's inequality, while in Sect. 3 we discuss the general case and establish our main result about "modified" logarithmic Sobolev and concentration inequalities under Poincaré inequalities. In the last part of this work, we collect several results on exponential integrability. In particular, we improve with this tool some recent upper bounds on spectral gaps (of Laplace operators for example) of F.R.K. Chung, A. Grigor'yan and S.-T. Yau [C-G-Y] in terms of distances between disjoint sets.

For simplicity, we denote throughout this work, for a function f on a probability space $(E, \mathscr{E}, \mu)$,

$$\mathrm{Var}_\mu(f) = \int f^2 \, d\mu - \left(\int f \, d\mu \right)^2$$

and, when $f \geqq 0$,

$$\mathrm{Ent}_\mu(f) = \int f \log f \, d\mu - \int f \, d\mu \log \int f \, d\mu$$

(under appropriate integrability conditions).

2 Logarithmic Sobolev inequality and Talagrand's concentration phenomenon for the exponential measure

We will work with the double exponential distribution ν_1. It is plain that all the results hold, with the obvious modifications, for the one-sided exponential distribution $\tilde{\nu}_1$. For the sake of completeness and comparison, we first recall the proof of (1.7). Denote by $\mathscr{L}_n$ the space of all continuous almost everywhere differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\int |f| \, d\nu_n < \infty$, $\int |\nabla f| \, d\nu_n < \infty$ and $\lim_{x_i \to \pm\infty} e^{-|x_i|} f(x_1, \dots, x_i, \dots, x_n) = 0$ for every $i = 1, \dots, n$ and $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \in \mathbb{R}$. The main argument of the proof is the following simple observation. If $\varphi \in \mathscr{L}_1$, by the integration by parts formula,

$$\int \varphi \, d\nu_1 = \varphi(0) + \int \operatorname{sgn}(x) \varphi'(x) \, d\nu_1 \, . \tag{2.1}$$

Lemma 2.1. *For every* $f \in \mathscr{L}_1$,

$$\operatorname{Var}_{\nu_1}(f) \leqq 4 \int f'^2 \, d\nu_1 \, .$$

Proof. Set $g(x) = f(x) - f(0)$. Then, by (2.1) and the Cauchy–Schwarz inequality,

$$\int g^2 \, d\nu_1 = 2 \int \operatorname{sgn}(x) g'(x) g(x) \, d\nu_1(x) \leqq 2 \left(\int g'^2 \, d\nu_1 \right)^{1/2} \left(\int g^2 \, d\nu_1 \right)^{1/2} \, .$$

Since $\operatorname{Var}_{\nu_1}(f) = \operatorname{Var}_{\nu_1}(g) \leqq \int g^2 \, d\nu_1$, and $g' = f'$, the lemma follows. □

We turn to the corresponding inequality for entropy (1.8) and the main result of this section.

Proposition 2.2. *For every Lipschitz function f on $\mathbb{R}$ such that $|f'| \leqq c < 1$ a.e.,*

$$\operatorname{Ent}_{\nu_1}(e^f) \leqq \frac{2}{1-c} \int f'^2 e^f \, d\nu_1 \, .$$

Note that Proposition 2.2, when applied to functions εf as $\varepsilon \to 0$, implies Lemma 2.1.

Proof. Changing f into $f + \text{const}$ we may assume that $f(0) = 0$. Since

$$u \log u \geqq u - 1, \quad u \geqq 0 \, ,$$

we have

$$\operatorname{Ent}_{\nu_1}(e^f) \leqq \int [f e^f - e^f + 1] \, d\nu_1 \, .$$

Since $|f'| \leqq c < 1$ a.e., the functions e^f, fe^f and $f^2 e^f$ all belong to $\mathscr{L}_1$. Therefore, by repeated use of (2.1),

$$\int [f e^f - e^f + 1] \, d\nu_1 = \int \operatorname{sgn}(x) f'(x) f(x) e^{f(x)} \, d\nu_1(x)$$

and

$$\begin{aligned} \int f^2 e^f \, d\nu_1 &= 2 \int \operatorname{sgn}(x) f'(x) f(x) e^{f(x)} \, d\nu_1(x) \\ &\quad + \int \operatorname{sgn}(x) f'(x) f(x)^2 e^{f(x)} \, d\nu_1(x) \, . \end{aligned}$$

By the Cauchy–Schwarz inequality and the assumption on f',

$$\int f^2 e^f \, d\nu_1 \leqq 2 \left(\int f'^2 e^f \, d\nu_1 \right)^{1/2} \left(\int f^2 e^f \, d\nu_1 \right)^{1/2} + c \int f^2 e^f \, d\nu_1$$

so that

$$\int f^2 e^f \, d\nu_1 \leqq \left(\frac{2}{1-c} \right)^2 \int f'^2 e^f \, d\nu_1 \, .$$

Now, by the Cauchy–Schwarz inequality again,

$$\begin{aligned} \mathrm{Ent}_{\nu_1}(e^f) &\leqq \int \mathrm{sgn}(x) f'(x) f(x) e^{f(x)} \, d\nu_1(x) \\ &\leqq \left(\int f'^2 e^f \, d\nu_1 \right)^{1/2} \left(\int f^2 e^f \, d\nu_1 \right)^{1/2} \leqq \frac{2}{1-c} \int f'^2 e^f \, d\nu_1 \end{aligned}$$

which is the result. Proposition 1 is established. □

We are now ready to describe the application to Talagrand's concentration inequality (1.4). The basic product property of entropy (cf. e.g. [Le3], Proposition 4.1) indicates that

$$\mathrm{Ent}_{\nu_n}(e^f) \leqq \int \sum_{i=1}^n \mathrm{Ent}_{\nu_1}(e^{f_i}) \, d\nu_n \tag{2.2}$$

where we write f_i to emphasise the fact that we consider f as a function of the i-variable, while all other coordinates are fixed. Thus, as a consequence of Proposition 2.2, for every smooth function f on $\mathbb{R}^n$ such that $\max_{1 \leqq i \leqq n} |\partial_i f| \leqq 1$ a.e. and every λ, $|\lambda| \leqq c < 1$,

$$\mathrm{Ent}_{\nu_n}(e^{\lambda f}) \leqq \frac{2\lambda^2}{1-c} \int \sum_{i=1}^n (\partial_i f)^2 e^{\lambda f} \, d\nu_n \, . \tag{2.3}$$

Let us take for simplicity $c = \frac{1}{2}$ (although $c < 1$ might improve some numerical constants below). Assume moreover that $\sum_{i=1}^n (\partial_i f)^2 \leqq \alpha^2$ a.e. and denote by $F(\lambda) = \int e^{\lambda f} d\nu_n$, $\lambda \in \mathbb{R}$, the Laplace transform of f. Then, by (2.3),

$$\mathrm{Ent}_{\nu_n}(e^{\lambda f}) = \lambda F'(\lambda) - F(\lambda) \log F(\lambda) \leqq 4\alpha^2 \lambda^2 F(\lambda)$$

for every $|\lambda| \leqq \frac{1}{2}$. Setting $H(\lambda) = (1/\lambda) \log F(\lambda)$, $H(0) = \int f \, d\nu_n$, shows that $H'(\lambda) \leqq 4\alpha^2$ for $|\lambda| \leqq \frac{1}{2}$. Therefore, always for $|\lambda| \leqq \frac{1}{2}$,

$$F(\lambda) = \int e^{\lambda f} \, d\nu_n \leqq e^{\lambda \int f \, d\nu_n + 4\alpha^2 \lambda^2} \, .$$

By Chebyshev's inequality, we finally get that

$$\nu_n \left(f \geqq \int f \, d\nu_n + t \right) \leqq \exp\left(-\frac{1}{4} \min\left(t, \frac{t^2}{4\alpha^2} \right) \right)$$

for every $t \geqq 0$, where f is thus a Lipschitz map on $\mathbb{R}^n$ with $\sum_{i=1}^n |\partial_i f|^2 \leqq \alpha^2$ and $\max_{1 \leqq i \leqq n} |\partial_i f| \leqq 1$ a.e. By homogeneity, this inequality amounts to (1.4) (with $K = 16$) and our claim is proved. As already mentioned, we have a similar result for the one-sided exponential measure.

To complete this section, let us sketch the equivalence between (1.3) and (1.4). (Although we present the argument for ν_n only, it of course extends to more general situations, as will be used in the next section.) To see that (1.3) implies (1.4), simply apply (1.3) to $A = \{f \leqq M\}$ where M is a median of f for ν_n and note that

$$A + \sqrt{r}B_2 + rB_1 \subset \{f \leqq M + \alpha\sqrt{r} + \beta r\} .$$

Using a routine argument (cf. [M-S], pp. 142–143), the deviation inequality (1.4) from either the median or the mean are equivalent up to numerical constants (with possibly a further constant in front of the exponential function). Now starting from (1.4) with M the mean for example, consider, for $A \subset \mathbb{R}^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$F_A(x) = \inf_{a \in A} \sum_{i=1}^n \min(|x_i - a_i|, |x_i - a_i|^2) .$$

For $r > 0$, set then $f = \min(F_A, r)$. We have $\sum_{i=1}^n |\partial_i f|^2 \leqq 4r$ and $\max_{1 \leqq i \leqq n} |\partial_i f| \leqq 2$ a.e. Indeed, it is enough to prove this result for $g = \min(g_a, r)$ for every fixed a where

$$g_a(x) = \sum_{i=1}^n \min(|x_i - a_i|, |x_i - a_i|^2) .$$

Now, a.e., and for every $i = 1, \ldots, n$, $|\partial_i g_a(x)| \leqq 2|x_i - a_i|$ if $|x_i - a_i| \leqq 1$ whereas $|\partial_i g_a(x)| \leqq 1$ if $|x_i - a_i| > 1$. Therefore, $\max_{1 \leqq i \leqq n} |\partial_i g_a(x)| \leqq 2$ and

$$\sum_{i=1}^n |\partial_i g_a(x)|^2 \leqq 4 \sum_{i=1}^n \min(|x_i - a_i|, |x_i - a_i|^2) = 4g_a(x)$$

which yields the result. Now, if $\nu_n(A) \geqq \frac{1}{2}$, $\int f \, d\nu_n \leqq r(1 - \nu_n(A)) \leqq r/2$. It then follows from (1.4) that

$$\nu_n(F_A \geqq r) = \nu_n(f \geqq r) \leqq \nu_n\left(f \geqq M + \frac{r}{2}\right) \leqq e^{-r/16K} .$$

Since $\{F_A \leqq r\} \subset A + \sqrt{r}B_2 + rB_1$, the claim follows.

3 The abstract case

In this section, we will investigate the preceding modified logarithmic Sobolev type inequalities (Proposition 2.2) in an abstract setting and will observe, somewhat surprisingly, that they always hold under a Poincaré inequality only. We may then prove concentration results similar to the one for the exponential distribution in a rather general framework.

Let us consider a probability measure μ say, for simplicity, on a metric space (E, d) equipped with its Borel sets. For a real-valued function f on E, we define the "length" of its gradient at $x \in E$ (possibly infinite) as

$$|\nabla f|(x) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)} .$$

We say that μ satisfies a Poincaré inequality with constant λ_1 if, for every f such that $\int f^2\,d\mu < \infty$ and $\int |\nabla f|^2\,d\mu < \infty$,

$$\lambda_1 \mathrm{Var}_\mu(f) \leqq \int |\nabla f|^2\,d\mu\,. \tag{3.1}$$

In what follows, we always assume that $\lambda_1 > 0$. We refer to [B-H] for more details on this (very) abstract setting (in which we do not want to enter here). Let us simply mention that such a framework should include Riemanian manifolds with finite (normalized Riemanian measure) for which λ_1 (with the usual notation) corresponds to the first eigenvalue of the Laplace operator. One may also consider Markov diffusion generators with the "carré du champ" operator [Ba]. We will be interested in this section in a modified logarithmic Sobolev inequality for measures μ for which $\lambda_1 > 0$ and its application to concentration properties for the product measures μ^n. The exponential integrability for μ itself will be studied in Sect. 4.

As announced, our main result indicates that we always have a logarithmic Sobolev inequality such as the one in Proposition 2.2 under a Poincaré inequality (3.1). More precisely, the following holds. For simplicity, we work with bounded functions. It is plain that the various results may appropriately be extended to larger classes of functions according to the setting with which we will be dealing. We write below $\mathrm{Var} = \mathrm{Var}_\mu$ and $\mathrm{Ent} = \mathrm{Ent}_\mu$.

Theorem 3.1. *For any bounded function f on E such that $|\nabla f| \leqq c < 2\sqrt{\lambda_1}$ μ-a.e.,*

$$\mathrm{Ent}(e^f) \leqq K(c) \int |\nabla f|^2 e^f\,d\mu\,,$$

where

$$K(c) = \frac{1}{2\lambda_1}\left(\frac{2\sqrt{\lambda_1}+c}{2\sqrt{\lambda_1}-c}\right)^2 e^{c\sqrt{5/\lambda_1}}\,.$$

As a corollary, we obtain, as in Sect. 2, a concentration inequality of Talagrand's type for the product measure μ^n of μ on E^n. As in (2.2) and (2.3), the logarithmic Sobolev inequality of the theorem tensorises to yield that, if f bounded on E^n satisfies $\max_{1\leqq i\leqq n}|\nabla_i f| \leqq 1$ a.e. with respect to the product measure μ^n (where $|\nabla_i f|$ denotes the length of the gradient with respect to the ith coordinate), then, for every $|\lambda| \leqq c < 2\sqrt{\lambda_1}$,

$$\begin{aligned}\mathrm{Ent}_{\mu^n}(e^{\lambda f}) &= \lambda \int f e^{\lambda f}\,d\mu^n - \int e^{\lambda f}\,d\mu^n \log \int e^{\lambda f}\,d\mu^n \\ &\leqq K(c)\lambda^2 \int \sum_{i=1}^n |\nabla_i f|^2 e^{\lambda f}\,d\mu^n\,.\end{aligned} \tag{3.2}$$

Integrating this differential inequality on the Laplace transform $\int e^{\lambda f}\,d\mu^n$ of f yields, as in Sect. 2, the following consequence.

Corollary 3.2. *Assume that μ satisfies* (3.1) *with $\lambda_1 > 0$ and denote by μ^n the product of μ on the product space E^n. Then, for every bounded function f on E^n such that*

$$\sum_{i=1}^n |\nabla_i f|^2 \leqq \alpha^2 \quad \text{and} \quad \max_{1\leqq i\leqq n} |\nabla_i f| \leqq \beta$$

μ-a.e., and every $t \geqq 0$

$$\mu^n\left(f \geqq \int f\, d\mu^n + t\right) \leqq \exp\left(-\frac{1}{K_1}\min\left(\frac{t}{\beta}, \frac{t^2}{\alpha^2}\right)\right),$$

where $K_1 > 0$ only depends on $\lambda_1 > 0$.

One may obtain a similar statement for products of possibly different measures μ with a uniform lower bound on the constants λ_1 in the Poincaré inequalities (3.1).

Following the argument at the end of Sect. 2, Corollary 3.2 may be turned into an inequality on sets such as (1.3). More precisely, if $\mu^n(A) \geqq \frac{1}{2}$, for every $r \geqq 0$ and some numerical constant $K > 0$,

$$\mu^n(F_A^h \geqq r) \leqq e^{-r/K}, \tag{3.3}$$

where $h(x, y) = \min(d(x, y), d(x, y)^2)$, $x, y \in E$, and, for $x = (x_1, \ldots, x_n) \in E^n$ and $A \subset E^n$,

$$F_A^h(x) = \inf_{a \in A} \sum_{i=1}^n h(x_i, a_i).$$

Inequalities such as (3.3) were considered by M. Talagrand in his study of general penalty functions h under conditions related to the measure μ [Ta3, Theorem 2.7.1]. In this special case however, his conditions on μ appear to be more stringent than a Poincaré inequality. Namely, if $E = \mathbb{R}$ and $h(x, y) = \min(|x - y|, |x - y|^2)$, conditions (2.7.1) and (2.7.2) of Theorem 2.7.1 in [Ta3] are equivalent to saying that, for every $r \geqq 0$,

$$\mu(A_r) \geqq G(G^{-1}(\mu(A))) + r,$$

where G is the distribution function of v_1 and G^{-1} is its inverse. This inequality is equivalent to its infinitesimal version as r tends to 0, that is

$$\mu^+(A) \geqq \min(\mu(A), 1 - \mu(A))$$

(where $\mu^+(A) = \limsup_{r \to 0}(1/r)[\mu(A_r) - \mu(A)]$). This may also be translated equivalently on functions as

$$\int |f - M|\, d\mu \leqq \int |\nabla f|\, d\mu$$

with M a median of f for μ (cf. [B-H]). In an abstract setting, and when h is defined as $h(x, y) = \min(d(x, y), d(x, y)^2)$, conditions (2.7.1) and (2.7.2) of [Ta3] thus similarly amounts to say that, for some $c > 0$ and every f,

$$c \int |f - M|\, d\mu \leqq \int |\nabla f|\, d\mu$$

with M a median of f for μ (or the mean actually). It is known that $\lambda_1 \geqq c^2/4$. This is the so-called Cheeger inequality in Riemanian geometry (if M is the mean, $\lambda_1 \geqq c^2/8$) ([Ch], [Bu], [Ya]). But the existence of $c > 0$ is in general a stronger condition than $\lambda_1 > 0$. For example, and following [Bu] (and the references therein), it is possible to perturb the Riemanian structure of a (compact) manifold near any given subdividing hypersurface as to make c

arbitrarily small with hardly affecting λ_1. One may also compare the two inequalities on the real line, with the usual gradient. On some interval containing the origin, say $[-1, +1]$, let μ be the probability measure with density $p(x) = \beta|x|^\alpha$, $0 < \alpha < 1$, where $\beta = (\alpha + 1)/2$ is the normalizing constant. Since $p(0) = 0$, $\mu^+([0, 1]) = 0$ while $\mu([0, 1]) = \frac{1}{2}$ so that the Cheeger constant c of μ is zero (cf. also [B-H]). On the other hand, this measure will satisfy a Poincaré inequality. Actually, if μ is any probability measure with density $p(x) > 0$ for almost every x in some interval I, then for every smooth function f on I,

$$\left(\int_I |f'(x)|\, dx\right)^2 \leqq \int_I |f'|^2\, d\mu \int_I \frac{1}{p(x)}\, dx \tag{3.4}$$

by the Cauchy–Schwarz inequality. Now, for any function f with values in $[a, b]$, and any measure μ, $\mathrm{Var}_\mu(f) \leqq \frac{1}{4}(b - a)^2$ so that, if f is smooth on I, and takes arbitrary values,

$$\mathrm{Var}_\mu(f) \leqq \frac{1}{4}\|f\|_{\mathrm{TV}}^2 = \frac{1}{4}\left(\int_I |f'(x)|\, dx\right)^2,$$

where $\|\cdot\|_{\mathrm{TV}}$ is the total variation norm. Together with (3.4), we thus get that if μ has density $p > 0$ on I,

$$\mathrm{Var}_\mu(f) \leqq C \int_I |f'^2|\, d\mu$$

with $C = \frac{1}{4}\int_I p(x)^{-1}\, dx$. When $p(x) = \beta|x|^\alpha$, $0 < \alpha < 1$, $C = 1/(1 - \alpha)^2 < \infty$ so that $\lambda_1 > 0$ for this measure while, as we have seen, $c = 0$. For this penalty function h, Theorem 3.1 thus improves, with a simple direct proof, Theorem 2.7.1 of [Ta3]. (When we informed M. Talagrand about this result, he mentioned to us that, while writing [Ta3], he convinced himself of the optimality of his conditions (2.7.1) and (2.7.2). As he now realizes it, the argument turned out to be erroneous and he can also modify the proof of his Theorem 2.7.1 in order to reach a similar conclusion.)

Before turning to the proof of Theorem 3.1, let us observe that for the case of the exponential measure ν, $\lambda_1 = \frac{1}{4}$ by Lemma 2.1 so that, for $c < 1$,

$$K(c) = 2\left(\frac{1 + c}{1 - c}\right)^2 e^{2\sqrt{5}c}$$

which is somewhat worse than the constant in Proposition 2.2. An important feature of this constant is however that $K(c) \to 1/(2\lambda_1)$ as $c \to 0$. In particular (and as in Proposition 2.2), the logarithmic Sobolev inequality of the theorem implies the Poincaré inequality (3.1) by applying it to functions εf with $\varepsilon \to 0$. On the other hand, let us consider the case of the canonical Gaussian measure γ_1 on the real line for which it is known that $\lambda_1 = 1$. Let g be a Lipschitz function on $\mathbb{R}$ and apply the multidimensional analogue (cf. (3.2)) of Theorem 3.1 to the functions

$$f(x) = g\left(\frac{x_1 + \cdots + x_n}{\sqrt{n}}\right), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n\,,$$

for which $\max_{1 \leqq i \leqq n} |\partial_i f| \leqq \|g\|_{\mathrm{Lip}}/\sqrt{n} = c_n < 2$ for n large enough. By the rotational invariance of Gaussian measures, and since $c_n \to 0$, we get in the limit

$$\mathrm{Ent}_{\gamma_1}(e^g) \leqq \frac{1}{2} \int g'^2 e^g \, d\gamma_1 ,$$

that is, Gross's logarithmic Sobolev inequality (1.5) with optimal constant. Therefore, for the Gaussian measure, Poincaré and logarithmic Sobolev inequalities are in a sense equivalent.

Proof of Theorem 3.1. We split the proof in two propositions of independent interest. We thus assume throughout the argument that (3.1) holds with $\lambda_1 > 0$.

Proposition 3.3. *For any bounded function* f *on* E *with* $|\nabla f| \leqq c < 2\sqrt{\lambda_1}$ *and* $\int f \, d\mu = 0$,

$$\int f^2 e^f \, d\mu \leqq \frac{1}{\lambda_1} \left(\frac{2\sqrt{\lambda_1} + c}{2\sqrt{\lambda_1} - c} \right)^2 \int |\nabla f|^2 e^f \, d\mu .$$

Proof. Set $a^2 = \int f^2 e^f \, d\mu$ and $b^2 = \int |\nabla f|^2 e^f \, d\mu$. By (3.1), for any two (bounded) functions g and h on E with $\int g \, d\mu = 0$,

$$\lambda_1^2 \left(\int gh \, d\mu\right)^2 \leqq \int |\nabla g|^2 \, d\mu \int |\nabla h|^2 \, d\mu .$$

Therefore,

$$4\lambda_1^2 \left(\int f e^{f/2} \, d\mu\right)^2 \leqq \int |\nabla f|^2 \, d\mu \int |\nabla f|^2 e^f \, d\mu \leqq c^2 b^2 . \tag{3.5}$$

In addition,

$$\lambda_1 \, \mathrm{Var}(f e^{f/2}) \leqq \int |\nabla f|^2 \left(1 + \frac{f}{2}\right)^2 e^f \, d\mu \leqq b^2 + \int |\nabla f|^2 f e^f \, d\mu + \frac{c^2 a^2}{4} .$$

By Cauchy–Schwarz,

$$\int |\nabla f|^2 f e^f \, d\mu = \int (|\nabla f| f e^{f/2})(|\nabla f| e^{f/2}) \, d\mu \leqq cab ,$$

so that

$$\lambda_1 \, \mathrm{Var}(f e^{f/2}) \leqq \left(b + \frac{ca}{2}\right)^2 . \tag{3.6}$$

Combining (3.5) and (3.6), we get that

$$a^2 = \left(\int f e^{f/2} \, d\mu\right)^2 + \mathrm{Var}(f e^{f/2}) \leqq \frac{c^2 b^2}{4\lambda_1^2} + \frac{1}{\lambda_1} \left(b + \frac{ca}{2}\right)^2 .$$

Hence, $(2\lambda_1 a)^2 \leqq (cb)^2 + \lambda_1 (2b + ca)^2$ so that $2\lambda_1 a \leqq cb + 2\sqrt{\lambda_1} b + c\sqrt{\lambda_1} a$ and

$$a(2\lambda_1 - c\sqrt{\lambda_1}) \leqq 2\sqrt{\lambda_1} + c$$

and the conclusion follows. Proposition 3.3 is established. $\square$

We turn to the second step of the proof.

Proposition 3.4. *For any bounded function f on E with $|\nabla f| \leqq c$ and $\int f\,d\mu = 0$, we have*

$$\int f^2\,d\mu \leqq e^{c\sqrt{5/\lambda_1}} \int f^2 e^{-|f|}\,d\mu\,.$$

Proof. For all $u > 0$ and $v \in \mathbb{R}$, we have $2|v| \leqq u + (1/u)v^2$, hence $2|v|^3 \leqq uv^2 + (1/u)v^4$. Therefore,

$$2\int |f|^3\,d\mu \leqq u\int f^2\,d\mu + \frac{1}{u}\int f^4\,d\mu\,. \tag{3.7}$$

Write $\int f^4\,d\mu = (\int f^2\,d\mu)^2 + \mathrm{Var}(f^2)$. By (3.1) and the hypothesis on $|\nabla f|$,

$$\lambda_1 \int f^2\,d\mu \leqq \int |\nabla f|^2\,d\mu \leqq c^2\,,$$

so that $\lambda_1(\int f^2\,d\mu)^2 \leqq c^2 \int f^2\,d\mu$. Again by (3.1), but now applied to f^2,

$$\lambda_1\,\mathrm{Var}(f^2) \leqq 4\int f^2|\nabla f|^2\,d\mu \leqq 4c^2 \int f^2\,d\mu\,.$$

It follows that $\lambda_1 \int f^4\,d\mu \leqq 5c^2 \int f^2\,d\mu$. According to (3.7), for every $u > 0$,

$$2\int |f|^3\,d\mu \leqq \left(u + \frac{5c^2}{u\lambda_1}\right)\int f^2\,d\mu\,.$$

Minimizing over $u > 0$ ($u = \sqrt{5c^2/\lambda_1}$), we get

$$\int |f|^3\,d\mu \leqq c\sqrt{\frac{5}{\lambda_1}}\int f^2\,d\mu\,. \tag{3.8}$$

Consider now the probability measure $d\tau = f^2\,d\mu/\int f^2\,d\mu$. Then, by Jensen's inequality,

$$\int f^2 e^{-|f|}\,d\mu = \int e^{-|f|}\,d\tau \int f^2\,d\mu \geqq e^{-\int |f|\,d\tau}\int f^2\,d\mu\,.$$

But, by (3.8),

$$\int |f|\,d\tau = \frac{\int |f|^3\,d\mu}{\int f^2\,d\mu} \leqq c\sqrt{\frac{5}{\lambda_1}}$$

from which the result follows. The proof of Proposition 3.4 is thus complete. □

We can now complete the proof of Theorem 3.1. Since both sides of the inequality we have to establish are invariant under the translations $f \to f + \mathrm{const}$, we may and do assume that $\int f\,d\mu = 0$. As in the proof of Proposition 2.2,

$$\mathrm{Ent}(e^f) \leqq \int [fe^f - e^f + 1]\,d\mu\,.$$

Since $\int f\,d\mu = 0$, by Taylor's formula,

$$\int [fe^f - e^f + 1]\,d\mu = \int \int_0^1 tf^2 e^{tf}\,dt\,d\mu\,.$$

Let $\varphi(t) = \int f^2 e^{tf}\, d\mu$ on [0,1]. By convexity, φ attains its maximum at either $t = 0$, or $t = 1$. By Proposition 3.4, and since $e^{-|f|} \leqq e^f$, $\varphi(0) \leqq e^{c\sqrt{5/\lambda_1}}\varphi(1)$, so that, for every $t \in [0,1]$, $\varphi(t) \leqq e^{c\sqrt{5/\lambda_1}}\varphi(1)$. It follows that

$$\mathrm{Ent}(e^f) \leqq \int_0^1 t\varphi(t)\, dt \leqq \int_0^1 te^{c\sqrt{5/\lambda_1}}\varphi(1)\, dt = \frac{1}{2}e^{c\sqrt{5/\lambda_1}} \int f^2 e^f\, d\mu\, .$$

Together with Proposition 3.3, Theorem 3.1 is established. □

4 Poincaré inequalities and exponential integrability

In the last part of this note, we present some results on exponential integrability under Poincaré type inequalities and on sharp upper bounds on spectral gaps using diameters. We take again the abstract (and informal) setting of Sect. 3 and assume that we have a Poincaré type inequality

$$\lambda_1 \,\mathrm{Var}(f) \leqq \int |\nabla f|^2\, d\mu \tag{4.1}$$

for every f, with $\lambda_1 > 0$.

It is known since the work [G-M] by M. Gromov and V.D. Milman that under (4.1), Lipschitz functions are exponentially integrable. (See also the work [B-U], the methods of which easily extend to an abstract setting as above.) More precisely, it is shown in [G-M], that, for every set A in E with $\mu(A) \geqq \frac{1}{2}$, and every $r \geqq 0$,

$$\mu(A_r) \geqq 1 - Ke^{-r\sqrt{\lambda_1}/K} \tag{4.2}$$

where $A_r = \{x \in E;\ d(x,A) < r\}$ and $K > 0$ is some numerical constant. Let now f be a Lipschitz function on E in the sense that $|\nabla f|(x) \leqq 1$ for μ-a.e. x in E. Let furthermore M be a median of f for μ, that is $\mu\ (f \geqq M)$, $\mu(f \leqq M) \geqq \frac{1}{2}$. Applying (4.2) to $A = \{f \leqq M\}$, and since $A_r \subset \{f \leqq M + r\}$, we deduce that, for every $t \geqq 0$,

$$\mu(f \geqq M + t) \leqq Ke^{-t\sqrt{\lambda_1}/K}\, .$$

Together with the corresponding inequality for $-f$, for every $t \geqq 0$,

$$\mu(|f - M| \geqq t) \leqq 2Ke^{-t\sqrt{\lambda_1}/K}\, .$$

In particular,

$$\int e^{\lambda|f-M|}\, d\mu \leqq 1 + \frac{2K^2\lambda}{\sqrt{\lambda_1} - K\lambda} \tag{4.3}$$

for every $\lambda < \sqrt{\lambda_1}/K$.

With different proofs, and somewhat improved bounds, this result was re-obtained in [A-M-S] and [A-S]. (It also follows from Corollary 3.2 or (3.3) with $n = 1$.) Moreover, sharp constants were recently deduced by M. Schmuckenschläger [Sc]. He showed that, under (4.1), for every function f such that $|\nabla f| \leqq 1$ and $\int f\, d\mu = 0$ and every $0 \leqq \lambda < 2\sqrt{\lambda_1}$,

$$\int e^{\lambda f}\, d\mu \leqq \frac{16\lambda_1^2}{(2\sqrt{\lambda_1} - \lambda)^4}\, . \tag{4.4}$$

The example of the exponential distribution ν_1, for which $\lambda_1 = \frac{1}{4}$, and the function $f(x) = x$, for which,

$$\int e^{\lambda f}\, d\nu_1 = \frac{1}{1-\lambda^2}$$

show that the condition $\lambda < 2\sqrt{\lambda_1}$ is sharp in (4.4). On the other hand, the bound on the integral in (4.4) is not sharp as $\lambda \to 2\sqrt{\lambda_1}$. This bound may be improved according to the next result. The proof is similar to the one in [A-S].

Proposition 4.1. *Assume that* (4.1) *holds and let f be such that $|\nabla f| \leqq 1$ and $\int f\, d\mu = 0$. Then, for every $0 \leqq \lambda < 2\sqrt{\lambda_1}$,*

$$\int e^{\lambda f}\, d\mu \leqq \frac{2\sqrt{\lambda_1}+\lambda}{2\sqrt{\lambda_1}-\lambda}\,.$$

Proof. Set $u(\lambda) = \int e^{\lambda f}\, d\mu$, $\lambda \geqq 0$, assuming for simplicity that f is bounded. Applying (4.1) to $e^{\lambda f/2}$ yields that $u(\lambda) - u(\lambda/2)^2 \leqq (\lambda^2/4\lambda_1)u(\lambda)$, that is, for every $\lambda < 2\sqrt{\lambda_1}$,

$$u(\lambda) \leqq \frac{4\lambda_1}{4\lambda_1-\lambda^2}\, u\left(\frac{\lambda}{2}\right)^2 .$$

Applying the same inequality for $\lambda/2$ and iterating, yields, after n steps,

$$u(\lambda) \leqq \prod_{k=0}^{n-1} \left(\frac{4\lambda_1}{4\lambda_1 - \lambda^2/4^k}\right)^{2^k} u\left(\frac{\lambda}{2^n}\right) .$$

Since $u(\lambda) = 1 + o(\lambda)$, we have that $u(\alpha\lambda)^\alpha \to 1$ as $\alpha \to \infty$. Therefore,

$$u(\lambda) \leqq U(\lambda) = \prod_{k=0}^{\infty} \left(\frac{4\lambda_1}{4\lambda_1 - \lambda^2/4^k}\right)^{2^k} ,$$

where the product converges whenever $\lambda < 2\sqrt{\lambda_1}$. The proof of the proposition is easily completed. Introduce

$$V(\lambda) = \prod_{k=1}^{\infty} \left(\frac{4\lambda_1}{4\lambda_1 - \lambda^2/4^k}\right)^{2^k}$$

so that

$$U(\lambda) = \frac{4\lambda_1}{4\lambda_1-\lambda^2}\, V(\lambda)\, .$$

It will be enough to show that

$$\sqrt{V(\lambda)} \leqq \frac{(2\sqrt{\lambda_1}+\lambda)}{2\sqrt{\lambda_1}}\,. \tag{4.5}$$

By Taylor's formula,

$$\begin{aligned} \log V(\lambda) &= \sum_{k=1}^{\infty} 2^k \log\left(\frac{1}{1-\frac{\lambda^2}{4\lambda_1}\cdot\frac{1}{4^k}}\right) \\ &= \sum_{k=1}^{\infty} 2^k \sum_{n=1}^{\infty} \frac{\lambda^{2n}}{n(4\lambda_1)^n}\cdot\frac{1}{4^{nk}} = \sum_{n=1}^{\infty}\left(\frac{\lambda^2}{4\lambda_1}\right)^n \frac{1}{n(2^{2n-1}-1)}\,. \end{aligned} \tag{4.6}$$

In particular, V is log-convex. Then $\sqrt{V(\lambda)}$ is convex and $V(0) = 1$ so that (4.5) amounts to say that $V(2\sqrt{\lambda_1}) \leqq 4$, that is

$$\begin{aligned}\log V(2\sqrt{\lambda_1}) &= \sum_{n=1}^{\infty} \frac{1}{n(2^{2n-1}-1)} \\ &\leqq 1 + \sum_{n=2}^{\infty} \frac{1}{n4^{n-1}} = 1 + 4\left(-\log\left(1-\frac{1}{4}\right) - \frac{1}{4}\right) = 4\log\frac{4}{3} .\end{aligned}$$

It remains to note that $(\frac{4}{3})^4 \leqq 4$. The proof of Proposition 4.1 is complete. $\square$

If we apply (4.1) to $\mathrm{sh}(\lambda f/2)$, when f is symmetrically distributed, we immediately get a somewhat sharper bound than the one in Proposition 4.1, namely

$$\int e^{\lambda f}\, d\mu \leqq \frac{4\lambda_1 + \lambda^2}{4\lambda_1 - \lambda^2}, \quad \lambda < 2\sqrt{\lambda_1} .$$

We will use the sh function in a similar argument below.

We now apply the preceding exponential integrability to bounds on the spectral gap λ_1 in terms of distances between disjoints sets. Let A, B be two disjoint sets in E with $D = d(A,B)$. Apply Proposition 4.1 (or (4.3)) to $f(x) = d(x,B)$ in the following way:

$$\int_A \int_B e^{\lambda(f(x)-f(y))}\, d\mu(x)\, d\mu(y) \leqq \int e^{\lambda(f-M)}\, d\mu \int e^{\lambda(M-f)}\, d\mu \leqq \left(\frac{2\sqrt{\lambda_1}+\lambda}{2\sqrt{\lambda_1}-\lambda}\right)^2$$

where $M = \int f\, d\mu$. Since, by the choice of f,

$$\int_A \int_B e^{\lambda(f(x)-f(y))}\, d\mu(x)\, d\mu(y) \geqq e^{\lambda D} \mu(A)\mu(B) ,$$

we obtain, taking for example $\lambda = \sqrt{\lambda_1}$,

$$\lambda_1 \leqq \frac{1}{D^2} \log^2\left(\frac{C}{\mu(A)\mu(B)}\right) \tag{4.7}$$

with $C = 9$ (starting with (4.3) would simply yield a worse constant C).

Inequalities such as (4.7) have been considered recently by F.R.K. Chung, A. Grigory'an and S.-T. Yau who showed (4.7) with $C = 4$ using heat kernel expansions [C-G-Y1], and then with $C = e$ [C-G-Y2] using the wave equation. (They actually establish similar inequalities for all the sequence of eigenvalues, something we do not consider here. They also establish similar results on graphs.) To conclude this work, we would like to briefly indicate how one may improve in a simple way, and in the spirit of the proof of Proposition 4.1 (and the methods developed in this work), the value of the numerical constant C in (4.7).

Start again with the Poincaré inequality (4.1), but on the product space $E \times E$ with product measure $\mu \otimes \mu$, that is

$$\lambda_1 \int f^2\, d\mu \otimes d\mu \leqq \int |\nabla_x f|^2 + |\nabla_y f|^2\, d\mu \otimes d\mu$$

for every f on $E \times E$ with $\int f \, d\mu \otimes d\mu = 0$. Apply this inequality to the function $f(x,y) = \mathrm{sh}(\lambda g(x,y)/2)$, $\lambda \geqq 0$, where $g(x,y) = h(x) - h(y)$ with $|\nabla h| \leqq 1$. Since $\int \mathrm{sh}(\lambda g/2) \, d\mu \otimes d\mu = 0$, we get

$$\lambda_1 \int \mathrm{sh}^2\left(\frac{\lambda g}{2}\right) d\mu \otimes d\mu \leqq \frac{\lambda^2}{4} \int |\nabla g|^2 \mathrm{ch}^2\left(\frac{\lambda g}{2}\right) d\mu \otimes d\mu \,,$$

that is

$$\int [4\lambda_1 - \lambda^2 |\nabla g|^2] \mathrm{ch}^2\left(\frac{\lambda g}{2}\right) d\mu \otimes d\mu \leqq 4\lambda_1 \,. \tag{4.8}$$

Now $|\nabla g|^2 \leqq 2$ (on the product space), so that, for every $\lambda < \sqrt{2\lambda_1}$, we already have

$$\int \mathrm{ch}^2\left(\frac{\lambda g}{2}\right) d\mu \otimes d\mu \leqq \frac{2\lambda_1}{2\lambda_1 - \lambda^2} \,.$$

By symmetry of $g(x,y) = h(x) - h(y)$ on $E \times E$,

$$\int \mathrm{ch}^2\left(\frac{\lambda g}{2}\right) d\mu \otimes d\mu = \frac{1}{2}\left(\int e^{\lambda g} d\mu \otimes d\mu + 1\right) \,.$$

Hence, for $\lambda < \sqrt{2\lambda_1}$,

$$\int e^{\lambda g} d\mu \otimes d\mu \leqq \frac{4\lambda_1}{2\lambda_1 - \lambda^2} - 1 = \frac{2\lambda_1 + \lambda^2}{2\lambda_1 - \lambda^2} \,. \tag{4.9}$$

Now, if $A, B \subset E$, $d(A,B) = D > 0$, and $h(x) = d(x,B)$, as before,

$$\int e^{\lambda g} \, d\mu \otimes d\mu \geqq \int_A \int_B e^{\lambda[d(x,B) - d(y,B)]} \, d\mu(x) \, d\mu(y) \geqq \mu(A)\mu(B) e^{\lambda D}.$$

Choose then $\lambda = \sqrt{\lambda_1}$ in (4.9) to get

$$\lambda_1 \leqq \frac{1}{D^2} \log^2\left(\frac{3}{\mu(A)\mu(B)}\right) \,.$$

It is not too difficult to improve the preceding argument. Start again with (4.8) and $g(x,y) = h(x) - h(y)$ and set $h(x) = \min(d(x,B), D)$. Then, as is easily seen,

- if $x, y \in A$, or $x, y \in B$, then $|\nabla g| = 0$, $g = 0$;
- if $x \in A$, $y \in B$ or $x \in B$, $y \in A$, then $|\nabla g| = 0$, $g = \pm D$;
- in any case, $|\nabla g|^2 \leqq 2$ and $\mathrm{ch}(\lambda g/2) \geqq 1$.

Then, if $0 < \lambda < \sqrt{2\lambda_1}$,

$$\begin{aligned} \int [4\lambda_1 - \lambda^2 |\nabla g|^2] \mathrm{ch}^2\left(\frac{\lambda g}{2}\right) d\mu \otimes d\mu \ \geqq \ & 4\lambda_1 [\mu(A)^2 + \mu(B)^2] \\ & + 8\lambda_1 \mathrm{ch}^2\left(\frac{\lambda D}{2}\right) \mu(A)\mu(B) \\ & + (4\lambda_1 - 2\lambda^2)[1 - (\mu(A) + \mu(B))^2] \,. \end{aligned}$$

Choosing $\lambda = \sqrt{\lambda_1}$ (but other choices are probably also interesting), we get together with (4.8),

$$8\mu(A)\mu(B)\mathrm{ch}^2\left(\frac{\sqrt{\lambda_1}D}{2}\right) \leqq 1 - (\mu(A) - \mu(B))^2 \leqq 1 \ ,$$

that is,

$$e^{\sqrt{\lambda_1}D} + e^{-\sqrt{\lambda_1}D} \leqq \frac{1}{\mu(A)\mu(B)} - 2 \leqq \frac{1}{\mu(A)\mu(B)} \ .$$

We may therefore state the following simple improvement upon [C-G-Y].

Proposition 4.2. *For any two sets A and B with $D = d(A,B)$,*

$$\lambda_1 \leqq \frac{1}{D^2}\log^2\left(\frac{1}{\mu(A)\mu(B)}\right) \ .$$

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