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ON MOMENTS OF POLYNOMIALS*

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Abstract. The equivalence of L^p-norms of polynomials of random variables is investigated.

Key words. moment inequalities, polynomials

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Let ξ be a random variable with nondegenerate distribution F_ξ on $\mathbf{R}$. For ξ normal, Yu. V. Prokhorov showed [3] that, for any polynomial $Q = Q(x)$, $x \in \mathbf{R}$, of degree $n \geqq 1$ with $\mathbf{E}Q(\xi) = 0$,

$$(\mathbf{E}|Q(\xi)|^2)^{1/2} \leqq c\,\mathbf{E}\,|Q(\xi)|, \tag{1}$$

where $c = c_n$ depends on n, only. He proved in [4] a similar inequality for ξ which has $\Gamma(\alpha)$-distribution with large parameter α ($\geqq 400n\log 2$). In the first case he used an expansion of Q in Hermite polynomials, and in the second paper, he used an expansion of Q in Laguerre polynomials. In this note, we suggest a simple argument to extend (1) to general r.v. ξ under some integrability assumptions. We show that the condition $\mathbf{E}Q(\xi) = 0$ is not necessary. We start with a more general inequality (which is stronger than (1) when $p \geqq 2$), namely,

$$\|Q(\xi)\|_p = (\mathbf{E}|Q(\xi)|^p)^{1/p} \leqq c\,\|Q(\xi)\|_0 \tag{2}$$

for the geometric mean $\|Q(\xi)\|_0 = \lim_{p\to 0^+} \|Q(\xi)\|_p = \exp \mathbf{E}\log|Q(\xi)|$ instead of the L^1-norm $\|Q(\xi)\|_1$ which is difficult to work with (all the constants $c = c_n(p, F_\xi)$ which we deal with may depend on n, p, F_ξ but not on Q). In contrast to the L^1-norm, one may use a general identity $\|f_1 \cdots f_n\|_0 = \|f_1\|_0 \cdots \|f_n\|_0$ which reduces (2) to polynomials of type $Q(x) = (x - z)^n$. More precisely, we have the following proposition.

PROPOSITION 1. *Let $\mathbf{E}|\xi|^{np} < \infty$, $(p > 0)$. The optimal constant c in* (2) *is given by*

$$c^{1/n} = \sup_{z\in\mathbf{C}} \frac{\|\xi - z\|_{np}}{\|\xi - z\|_0}. \tag{3}$$

In the case $c < \infty$ (and only in this case), there exist positive constants a and b such that for any polynomial $Q(x) = \prod_{i=1}^n (x - z_i)$, $z_i \in \mathbf{C}$ $(1 \leqq i \leqq n)$, and for all $r \in [0, p]$,

$$a\prod_{i=1}^n \big(1 + |z_i|\big) \leqq \|Q(\xi)\|_r \leqq b\prod_{i=1}^n \big(1 + |z_i|\big). \tag{4}$$

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Remark 2. Let $\mathbf{E}|\xi|^{np} < \infty$. The constant c defined by (3) is finite if and only if

$$\sup_{a\in\mathbf{R}} \int_0^1 \frac{\mathbf{P}\{|\xi - a| < t\}}{t}\, dt < +\infty. \tag{5}$$

Proof. Clearly, for some $c_1, c_2 > 0$, we have

$$c_1\bigl(1 + |z|\bigr) \leqq \|\xi - z\|_{np} \leqq c_2\bigl(1 + |z|\bigr),$$

whenever $z \in \mathbf{C}$. Hence, $c < \infty$ holds if and only if $\inf_z \|\xi - z\|_0/(1 + |z|) > 0$. This infimum may be taken over all real $z = a$. Indeed, writing $z = a + bi$, $a, b \in \mathbf{R}$ and assuming that $\|\xi - a\|_0 \geqq c_3(1 + |a|)$, we have

$$\begin{aligned}\|\xi - z\|_0 &= \bigl\|(\xi - a)^2 + b^2\bigr\|_0^{1/2} \geqq \bigl(\|\xi - a\|_0^2 + b^2\bigr)^{1/2}\\ &\geqq \bigl(c_3^2(1 + |a|)^2 + b^2\bigr)^{1/2} \geqq c_4(1 + |z|),\end{aligned}$$

where we used the property $\|f + g\|_0 \geqq \|f\|_0 + \|g\|_0$ for f and g non-negative with $f = (\xi - a)^2$ and $g = b^2$.

Now, write the relation $\inf_{a\in\mathbf{R}} \|\xi - a\|_0/(1 + |a|) > 0$ in the form

$$\mathbf{E}\log|\xi - a| \geqq D + \log(1 + |a|), \quad a \in \mathbf{R}, \tag{6}$$

for some $D \in \mathbf{R}$. Since $\mathbf{E}\log(1 + |\xi|) < +\infty$, we get, for all $a \in \mathbf{R}$, obvious estimates

$$\begin{aligned}&\bigl|\mathbf{E}\log(1 + |\xi - a|) - \log(1 + |a|)\bigr| \leqq \mathbf{E}\log(1 + |\xi|),\\ &\mathbf{E}\log(1 + |\xi - a|) - \mathbf{E}\log|\xi - a|\,\mathbf{1}_{\{|\xi - a|\geqq 1\}} \leqq \log 2,\end{aligned}$$

from which it follows that

$$\Bigl|\mathbf{E}\log|\xi - a|\,\mathbf{1}_{\{|\xi - a|\geqq 1\}} - \log\bigl(1 + |a|\bigr)\Bigr| \leqq D', \quad a \in \mathbf{R},$$

for some $D' \in \mathbf{R}$. Therefore, (6) may be written as

$$-\mathbf{E}\log|\xi - a|\,\mathbf{1}_{\{|\xi - a|<1\}} \leqq D'', \quad a \in \mathbf{R},$$

for some $D'' \in \mathbf{R}$, which is exactly (5).

For example, when the distribution ξ is unimodal, with mode a_0, we have for all a,

$$\mathbf{P}\bigl\{|\xi - a| < t\bigr\} \leqq \mathbf{P}\bigl\{|\xi - a_0| < 2t\bigr\},$$

and (5) becomes

$$\int_0^1 \frac{\mathbf{P}\{|\xi - a_0| < t\}}{t}\, dt < \infty.$$

In particular, this condition is satisfied for $\Gamma(\alpha)$-distribution with arbitrary $\alpha > -1$. It should be noted, however, that the method used by Prokhorov yields constants c_n in (1) which increase exponentially (as functions of n), while in the case $r = 0$, we get a worse rate in (4) unless ξ is bounded: in this case, according to (3), the sequence $c^{1/n} = c_n(p, F_\xi)^{1/n}$ tends to infinity.

Proof of Proposition 1. Writing $Q(x) = A\prod_{i=1}^n (x - z_i)$, $x \in \mathbf{C}$, and defining c as in (3), we get, by Hölder's inequality,

$$\|Q(\xi)\|_p \leqq |A| \prod_{i=1}^n \|\xi - z_i\|_{np} \leqq c\,|A| \prod_{i=1}^n \|\xi - z_i\|_0 = c\,\|Q(\xi)\|_0.$$

Thus, (2) is proved. Similarly, it suffices to consider in (4) only polynomials of the form $Q(x) = (x-z)^n$. In addition, the cases $r = 0$ for the left inequality and $r = p$ for the right inequality have to be considered, only. The second case, i.e., the inequality $\|\xi - z\|_{np} \leqq \text{const.}\,(1 + |z|)$ holds due to the assumption $\|\xi\|_{np} < \infty$. The first case, i.e., the inequality $\|\xi - z\|_0 \geqq \text{const.}\,(1 + |z|)$ is equivalent to $c < \infty$ (as explained in Remark 2).

Remark 3. It would be very interesting to know, whether or not inequalities between L^p- and L^q-norms like (1) hold for polynomials Q in d random identically distributed random variables $\xi = (\xi_1, \dots, \xi_d)$. One may expect that, under a general assumption such as integrability, such a conjecture is valid, and the constants are independent of the dimension d. This is true, for example, for random variables ξ_i which are uniformly distributed in an interval, and thus, by the central limit theorem, for normally distributed ξ_i as well (cf. [2] for a more general result). Some related Khinchin–Kahane-type inequalities for Bernoullian random variables may be found in [1] and [5].

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