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ISOPERIMETRIC AND ANALYTIC INEQUALITIES FOR LOG-CONCAVE PROBABILITY MEASURES[1]

BY S. G. BOBKOV

Syktyvkar University

We discuss an approach, based on the Brunn–Minkowski inequality, to isoperimetric and analytic inequalities for probability measures on Euclidean space with logarithmically concave densities. In particular, we show that such measures have positive isoperimetric constants in the sense of Cheeger and thus always share Poincaré-type inequalities. We then describe those log-concave measures which satisfy isoperimetric inequalities of Gaussian type. The results are precised in dimension 1.

1. Introduction. It is often useful to know whether or not a given probability measure μ on Euclidean space $\mathbf{R}^n$ satisfies certain isoperimetric or analytic inequalities. Of most interest are the isoperimetric inequalities of Cheeger and Gaussian types,

$$\mu^+(A) \geq c \min\{\mu(A), 1 - \mu(A)\}, \tag{1.1}$$

$$\mu^+(A) \geq c\varphi(\Phi^{-1}(\mu(A))). \tag{1.2}$$

Here $c > 0$, Φ^{-1} is the inverse of the normal distribution function Φ with density $\Phi'(x) = \varphi(x) = (1/\sqrt{2\pi})\exp(-x^2/2)\ (x \in \mathbf{R})$, A is an arbitrary measurable set in $\mathbf{R}^n$ and $\mu^+(A)$ denotes the μ-perimeter of A defined by

$$\mu^+(A) = \liminf_{h \to 0^+} \frac{\mu(A^h) - \mu(A)}{h},$$

where $A^h = \{x \in \mathbf{R}^n \colon \exists a \in A,\ |x - a| < h\}$ is an h-neighborhood of A. The function $I(p) = \varphi(\Phi^{-1}(p))$ appearing in (1.2) is the so-called Gaussian isoperimetric function: for the canonical Gaussian measure $\mu = \gamma_n$ on $\mathbf{R}^n$, (1.2) holds with $c = 1$ for all n and represents a form of the Gaussian isoperimetric inequality (cf. [13], [32]). Inequality (1.1) is weaker and is known to hold for a large family of product probability measures. For example, with a dimension free constant $c > 0$, it is satisfied by the products $\mu = \nu^n$ of the two-sided exponential distribution with density $d\nu(x)/dx = \frac{1}{2}e^{-|x|}$, $x \in \mathbf{R}$ (cf. [11]).

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In general, there is a probabilistic way to express (1.1) and (1.2): with respect to μ, the distribution of every Lipschitz function on $\mathbf{R}^n$ with Lipschitz seminorm less than or equal to 1 represents a Lipschitz image of measure ν (resp., γ_1), with Lipschitz seminorm less than or equal to $1/c$. These inequalities may equivalently be written in a functional form as certain analytic inequalities. In particular, (1.1) and (1.2) imply, respectively, that, for every smooth (or even locally Lipschitz) function f on $\mathbf{R}^n$ with gradient ∇f,

$$\lambda_1 \operatorname{Var}_\mu(f) \le \int |\nabla f|^2 \, d\mu, \tag{1.3}$$

$$\frac{\rho}{2} \operatorname{Ent}_\mu(f^2) \le \int |\nabla f|^2 \, d\mu. \tag{1.4}$$

Here $\operatorname{Var}_\mu(f) = \int f^2 \, d\mu - (\int f \, d\mu)^2$ denotes the variance of f, and

$$\operatorname{Ent}_\mu(f^2) = \int f^2 \log f^2 \, d\mu - \int f^2 \, d\mu \log \int f^2 \, d\mu$$

denotes the entropy of f^2 with respect to μ. As first shown in 1970 by Cheeger [17], Poincaré-type inequalities (1.3) hold with $\lambda_1 = c^2/4$, and the optimal value of $c = Is(\mu)$ in (1.1) is now often referred to as the (Cheeger) isoperimetric constant of μ. The second, logarithmic Sobolev inequalities (1.4), were introduced in 1975 by Gross [20], and later Ledoux connected them with (1.2); the logarithmic Sobolev constant ρ of μ satisfies $\rho \ge Kc^2$ for some universal K ([24]).

In this note, we consider the above isoperimetric and analytic inequalities in the case of an absolutely continuous log-concave probability measure μ on $\mathbf{R}^n$. By definition, this means that μ has a density $d\mu(x)/dx = \exp(-V(x))$, $x \in \mathbf{R}^n$, with an arbitrary convex function $V\colon \mathbf{R}^n \to (-\infty, +\infty]$. In addition to convexity, the only requirement on V is that $\mu(\mathbf{R}^n) = \int \exp(-V(x)) \, dx = 1$. As a main result, we present an inequality which relates μ-perimeter of sets to their size and to μ-distribution of the Euclidean norm.

THEOREM 1.1. *Let μ be a log-concave probability measure on $\mathbf{R}^n$. For all measurable sets $A \subset \mathbf{R}^n$, for every point $x_0 \in \mathbf{R}^n$ and every number $r > 0$,*

$$\begin{aligned} 2r\mu^+(A) \ge \mu(A)\log\frac{1}{\mu(A)} + (1-\mu(A))\log\frac{1}{1-\mu(A)} \\ + \log \mu\{|x - x_0| \le r\}. \end{aligned} \tag{1.5}$$

Inequality (1.5) will be derived from the Brunn–Minkowski inequality in the Prékopa–Leindler functional form. If one takes x_0 to be the barycenter of μ and $r = r(\mu(A))$ to be sufficiently large, the sum of the last two terms in (1.5) will be positive. This leads to an isoperimetric inequality

$$\mu^+(A) \ge I(\mu(A)) \quad \text{with } I(p) = \frac{p}{2r(p)} \log\frac{1}{p}, \tag{1.6}$$

that is, with a function $I(p)$ whose behavior near zero is determined by behavior of the tails $\mu\{|x-x_0|>r\}$ for large r. In such a form, (1.6) unites a few known results obtained by different methods and in different contexts. First, in the worst case, by the well-known Borell's theorem, the tails $\mu\{|x-x_0|>r\}$ have exponential decreasity, and (1.6) gives an inequality of Cheeger type together with a bound for the isoperimetric constant:

THEOREM 1.2. *For every log-concave probability measure μ on $\mathbf{R}^n$,*

$$Is(\mu) \geq \frac{1}{K\| |x-x_0| \|_{L_2(\mu)}}, \tag{1.7}$$

where K is a universal constant, and $x_0 = \int x\, d\mu(x)$ is barycenter of μ.

Here $\| |x-x_0| \|_{L^2(\mu)} = (\int |x-x_0|^2\, d\mu(x))^{1/2}$ denotes the $L^2(\mu)$-norm of the function $x \to |x-x_0|$. Theorem 1.2 is due to Kannan, Lovász and Simonovits [22] proving (1.7), in an equivalent form, with the help of the so-called localization method (cf. [28]). They considered the case where μ is a uniform distribution on an arbitrary convex set, but their proof covers actually the general log-concave case.

In particular, Poincaré-type inequalities (1.3) hold for all log-concave measures μ, and, by Cheeger's theorem, we have an estimate,

$$\lambda_1 \geq \frac{1}{4K^2\| |x-x_0| \|^2_{L^2(\mu)}}. \tag{1.8}$$

This property cannot, however, be improved by replacing λ_1 with ρ, since the logarithmic Sobolev inequality (1.4) requires that μ possesses a stronger integrability property: $\int \exp(\varepsilon|x|^2)\, d\mu(x) < +\infty$, for some $\varepsilon > 0$. This is the well-known Herbst necessary condition. We will show, again on the basis of (1.5) and (1.6), that this condition is also sufficient in order that log-concave measures satisfy (1.2) and (1.4).

THEOREM 1.3. *The following properties are equivalent*:

(a) *μ satisfies* (1.2) *for some $c > 0$;*
(b) *μ satisfies* (1.4) *for some $\rho > 0$;*
(c) $\int \exp(\varepsilon|x|^2)\, d\mu(x) < +\infty$ *for some $\varepsilon > 0$.*

In addition, for the optimal value of ρ in (1.4), *we have*

$$\rho \geq \frac{1}{K\| |x-x_0| \|^2_{L^\psi(\mu)}}, \tag{1.9}$$

where K is universal, x_0 is barycenter of μ and where $\|\cdot\|_{L^\psi(\mu)}$ denotes the Orlicz norm generated by the Young function $\psi(u) = \exp(u^2) - 1$.

Recall that $\| |x-x_0| \|_{L^\psi(\mu)} = \inf\{t > 0\colon \int \exp(|x-x_0|^2/t^2)\, d\mu(x) \leq 2\}$.

(b) $\Rightarrow$ (a) was proved by Bakry and Ledoux in an abstract framework of Markov diffusion generators and semigroups ([2], Section 4). Actually, the equivalence of (a)–(c) can be shown as an application of the semigroup approach even for a bigger class of densities. Namely, this equivalence remains to hold in the case $V'' \geq c\,\mathrm{Id}$, $c \in \mathbf{R}$, where V'' is Hessian, that is, the matrix of the second derivatives of V (cf. [27]). Under this condition and in a more general setting, (c) $\Rightarrow$ (b) was first proved by Wang [33].

All the estimates (1.7)–(1.9) may be reversed provided that the function $\xi(x) = |x - x_0|$ is replaced with $\xi - \mathbf{E}\xi$ where $\mathbf{E}\xi$ is the expectation (the average value) of ξ over μ. Thus, we have, for example,

$$\frac{1}{K_0 \|\xi\|^2_{L^2(\mu)}} \leq \lambda_1 \leq \frac{1}{\|\xi - \mathbf{E}\xi\|^2_{L^2(\mu)}},$$

with some universal K_0. There can be, however, a big gap between the left- and right-hand sides of this inequality that may easily be seen in spaces of large dimensions. It was conjectured in [22] that, for any uniform distribution μ on a convex set, the value of $Is(\mu)$ is, up to an absolute (dimension free) constant, just an optimal value of c in the inequality (1.1) restricted to the class of all linear functions. There is a good reason to believe that the same holds in the general log-concave case and concerns also the constant λ_1 in the Poincaré-type inequality (1.3). If true, this would be a very remarkable observation, providing in particular a natural generalization of the concentration property of product measures to the case of dependent noncorrelated coordinates. Behind the class of product measures, sharp estimates for λ_1 are of course known in many important situations. For example, for every log-concave probability measure μ on $\mathbf{R}^n$ such that V is everywhere twice continuously differentiable with a positively definite Hessian, we have a remarkable Brascamp–Lieb's inequality [16]

$$\mathrm{Var}_\mu(f) \leq \int \left\langle \nabla f(x), V''(x)^{-1} \nabla f(x) \right\rangle d\mu(x),$$

where $V''(x)^{-1}$ is the inverse of $V''(x)$. In the case $V'' \geq c\,\mathrm{Id}$, $c > 0$, it implies that $\lambda_1 \geq c$. In particular, this gives the Gaussian Poincaré-type inequality, that is, (1.3) for γ_n with the best constant $\lambda_1 = 1$. The latter property was sharpened in [2], proving under the same hypothesis $V'' \geq c\,\mathrm{Id}$ the stronger inequality (1.2). This provides already a certain generalization of the Gaussian isoperimetric inequality.

The organization of the paper is as follows.

Theorem 1.1 is proved in Section 2. Here we also consider log-concave measures with a compact support, in which case (1.5) becomes an isoperimetric inequality which is somewhat sharper than (1.2) [in the sense of small values of $\mu(A)$]. As an application, we obtain a lower estimate for ρ in terms of λ_1 very similar to the one presented recently in [27] in the context of Riemannian geometry.

Theorems 1.1 and 1.2 are proved in Section 3.

Section 4 is devoted to the one-dimensional case; here we show that the estimates (1.7)–(1.9) are sharp. Thus, the logarithmic Sobolev constant ρ of μ turns out to be roughly $\| |x - x_0| \|_{L^{\psi}(\mu)}^{-2}$. This simplifies a description of ρ obtained by different tools in [10] in the class of all probability measures on $\mathbf{R}$. We discuss also a connection of two-sided estimates for the Cheeger isoperimetric constant $Is(\mu)$ with the Hensley theorem on the ratio of volumes of $(n-1)$-dimensional sections of convex isotropic bodies in $\mathbf{R}^n$. We will observe that this theorem, with the optimal constant of Ball [4], still holds for sets without a point of symmetry.

In the last section, we study some related problems about distribution of norms under log-concave measures.

2. Proof of Theorem 1.1. Compactly supported log-concave measures.

Given a symmetric bounded convex set $B \subset \mathbf{R}^n$ with the nonempty interior, define an associate norm $\|x\|_B = \inf\{t > 0\colon\ x \in tB\}$, and the dual norm

$$\|x\|_{B^\circ} = \sup_{b \in B} \langle x, b \rangle, \qquad x \in \mathbf{R}^n,$$

where $\langle \cdot , \cdot \rangle$ is the usual scalar product. Set $B(x_0) = x_0 + B$. Inequality (1.5) is a particular case of the following functional inequality.

PROPOSITION 2.1. *For every locally Lipschitz function f on $\mathbf{R}^n$ with values in* $[0,1]$, *and every* $x_0 \in \mathbf{R}^n$,

$$2\int \|\nabla f(x)\|_{B^\circ}\, d\mu(x) \geq \mathrm{Ent}_\mu(f) + \mathrm{Ent}_\mu(1-f) + \log \mu(B(x_0)). \tag{2.1}$$

PROOF. The argument is close to the one used by B. Maurey [29] to get a sharp concentration inequality for the Gaussian measure. First of all, we may assume that f is smooth, is constant outside a compact set and is such that $0 < f(x) < 1$, for all $x \in \mathbf{R}^n$. By Prékopa–Leindler's theorem (cf., e.g., [16], [30]), for all $t, s > 0$ with $t + s = 1$, and for all nonnegative measurable functions u, v, w on $\mathbf{R}^n$ such that $w(tx + sy) \geq u(x)^t v(y)^s$ whenever $x, y \in \mathbf{R}^n$, we have

$$\int w(z)\, dz \geq \left(\int u(x)\, dx\right)^t \left(\int v(y)\, dy\right)^s.$$

Applying this inequality to

$$u(x) = f(x)^{1/t} \exp(-V(x)), \qquad v(y) = 1_{B(x_0)}(y) \exp(-V(y)),$$
$$w(z) = f_t(z) \exp(-V(z))$$

(where $1_{B(x_0)}$ is the indicator function of the set $B(x_0)$), we get

$$\int f_t\, d\mu \geq \left(\int f^{1/t}\, d\mu\right)^t \mu(B(x_0))^s, \tag{2.2}$$

provided the function f_t satisfies $f_t(tx+sy) \geq f(x) 1_{B(x_0)}(y)$, for all $x, y \in \mathbf{R}^n$. In particular, we can take

$$f_t(z) = \sup_{b \in B} f\left(z + \frac{s}{t}(z - b - x_0)\right).$$

For s small,

$$f_t(z) = f(z) + \left[\|\nabla f(z)\|_{B^\circ} + \langle \nabla f(z), z - x_0 \rangle\right] s + O(s^2)$$

uniformly over all $z \in \mathbf{R}^n$. Letting in (2.2) $s \to 0$, we then arrive at the following inequality:

$$\int \|\nabla f(x)\|_{B^\circ} \, d\mu(x) + \int \langle \nabla f(x), x - x_0 \rangle \, d\mu(x)$$

$$\geq \mathrm{Ent}_\mu(f) + \int f \, d\mu \log \mu(B(x_0)).$$

Summing it with the same inequality for the function $1-f$, we obtain (2.1).

When $B = B_r$ is a Euclidean ball with center 0 and radius r, $\|\nabla f(x)\|_{B^\circ} = r|\nabla f(x)|$, and (2.1) becomes

$$2r \int |\nabla f| \, d\mu(x) \geq \mathrm{Ent}_\mu(f) + \mathrm{Ent}_\mu(1-f) + \log \mu(B_r(x_0)). \tag{2.3}$$

One can further approximate the indicator function of a measurable set $A \subset \mathbf{R}^n$ by Lipschitz functions with values in $[0,1]$, and then (2.3) yields (1.5),

$$\mu^+(A) \geq \frac{\mu(A)\log(1/\mu(A)) + (1-\mu(A))\log(1/1-\mu(A)) + \log \mu(B_r(x_0))}{2r}, \tag{2.4}$$

which holds for all $x_0 \in \mathbf{R}^n$ and $r > 0$. Choosing somewhat optimal r with x_0 the barycenter of μ, we will reach in the next section the isoperimetric inequalities of the form (1.1) and (1.2). Now let us look at the particular case of a compactly supported measure. Applying (2.3) and (2.4) to the ball $B_r(x_0)$ supporting the measure μ, we get one corollary.

COROLLARY 2.2. *Assume μ is concentrated in an Euclidean ball of radius r. Then, for every locally Lipschitz function f on $\mathbf{R}^n$ with values in* $[0,1]$, *and for every measurable set $A \subset \mathbf{R}^n$*,

$$2r \int |\nabla f| \, d\mu \geq \mathrm{Ent}_\mu(f) + \mathrm{Ent}_\mu(1-f), \tag{2.5}$$

$$2r\mu^+(A) \geq \mu(A) \log \frac{1}{\mu(A)} + (1-\mu(A)) \log \frac{1}{1-\mu(A)}. \tag{2.6}$$

It is of course very natural that these inequalities are stronger than (1.1)–(1.4). For example, (2.6) implies $\mu^+(A) \geq (\log 2/r) \min\{\mu(A), 1-\mu(A)\}$, so that

$$Is(\mu) \geq \frac{\log 2}{r}. \tag{2.7}$$

In particular, by Cheeger's inequality,

$$\lambda_1 \geq \frac{\log^2 2}{4r^2}. \tag{2.8}$$

In connection with randomized volume algorithms, inequalities similar to (2.7), in terms of diameter of the convex set supporting μ, were successively and by different methods studied by a number of authors (cf. [22]).

Another interesting feature of inequalities of the form (2.5) and (2.6), with μ being even an arbitrary probability measure on a metric space M, is that they show, in terms of r, a certain equivalence between the spectral gap λ_1 and the logarithmic Sobolev constant ρ, the optimal constants in (1.3) and (1.4). Indeed, in general, $\rho \leq \lambda_1$. Now, starting from (2.5) and following an argument of Rothaus [31], apply this inequality to f^2. We have, by the Schwarz inequality,

$$\begin{aligned} \operatorname{Ent}_\mu(f^2) &\leq 2r\int |\nabla f^2|\, d\mu = 4r\int |f|\,|\nabla f|\, d\mu \\ &\leq 4r\left(\int f^2\, d\mu\right)^{1/2} \left(\int |\nabla f|^2\, d\mu\right)^{1/2}. \end{aligned}$$

If $\int f\, d\mu = 0$, then $\int f^2\, d\mu \leq (1/\lambda_1)\int |\nabla f|^2\, d\mu$; hence,

$$\operatorname{Ent}_\mu(f^2) \leq \frac{4r}{\sqrt{\lambda_1}} \int |\nabla f|^2\, d\mu.$$

In general, by Rothaus' inequality $\operatorname{Eng}_\mu(f^2) \leq \operatorname{Ent}_\mu((f - \int f\, d\mu)^2) + 2\operatorname{Var}_\mu(f)$ (cf. [31], Lemma 9), we finally conclude that

$$\operatorname{Ent}_\mu(f^2) \leq \left(\frac{4r}{\sqrt{\lambda_1}} + \frac{2}{\lambda_1}\right) \int |\nabla f|^2\, d\mu.$$

We can now summarize

COROLLARY 2.3. *If μ is concentrated in an Euclidean ball of radius r, then*

$$\frac{\lambda_1}{1 + 2r\sqrt{\lambda_1}} \leq \rho \leq \lambda_1. \tag{2.9}$$

In particular, by virtue of (2.8),

$$\rho \geq \frac{1}{10r^2}. \tag{2.10}$$

Thus, we have arrived exactly at the same left estimate in (2.9) and, up to the constant, at the same estimate (2.10), which have recently been proved (after a prior contribution by Wang [33]) by Bakry, Ledoux and Qian [3] for a compact Riemannian manifold M with nonnegative Ricci curvature, of diameter $2r$, and with its Riemannian measure μ (cf. [3], [27], Theorem 7.3, and also [31], Theorem 10, for a similar inequality). The left estimate in (2.9) can be shown to be sharp for the measures μ_k on $[0,1]$ with densities $d\mu_k(x)/dx = (k+1)x^k$ $(k \geq 1)$. In this case, λ_1 is of order k^2 while ρ is of order k.

3. Proof of Theorems 1.2 and 1.3. We may and do assume that $x_0 = 0$. In the proof of Theorem 1.1, we will optimize (2.4) using an inequality due to Borell (cf. [12], Lemma 3.1).

LEMMA 3.1. *For all $t \geq 1$ and for every convex set $B \subset \mathbf{R}^n$, symmetric about the origin, with $\theta = \mu(B) > \frac{1}{2}$,*

$$1 - \mu(tB) \leq (1-\theta)\left(\frac{1-\theta}{\theta}\right)^{(t-1)/2}. \tag{3.1}$$

PROOF OF THEOREM 1.2. We apply the above estimate to the Euclidean ball $B = B_r$ with r such that $\mu(B_r) = \frac{2}{3}$, so that $((1-\theta)/\theta) = \frac{1}{2}$. Since for $x \in (0,1)$, $-\log(1-x) \leq x/(1-x)$, we get, by (3.1),

$$-\log \mu(tB_r) \leq -\log\left(1 - \tfrac{1}{3}2^{-(t-1)/2}\right) \leq 2^{-(t+1)/2}.$$

Therefore, according to (2.4) with tr (instead of r), for every measurable set $A \subset \mathbf{R}^n$ of measure $p = \mu(A)$,

$$\begin{aligned} \mu^+(A) &\geq \frac{p\log(1/p) + (1-p)\log(1/(1-p)) + \log \mu(tB_r)}{2rt} \\ &\geq \frac{p\log(1/p) + (1-p)\log(1/(1-p)) - 2^{-(t+1)/2}}{2rt}. \end{aligned} \tag{3.2}$$

Assume $0 < p \leq \frac{1}{2}$ and apply (3.2) to $t = 3\log(1/p)$. Then, $t \geq 1$, for all $p \in (0, \frac{1}{2}]$, and the second term $(1-p)\log(1/(1-p))$ in the right-hand side of (3.2) majorizes the third one $2^{-(t+1)/2}$. To see this, consider the function

$$u(p) = (1-p)\log\frac{1}{1-p} - 2^{-(t+1)/2} = (1-p)\log\frac{1}{1-p} - \frac{1}{\sqrt{2}}p^{3\log 2/2}.$$

This function is concave on $[0, \frac{1}{2}]$, and $u(0) = 0$. Hence, u will be nonnegative, if $u(\frac{1}{2}) \geq 0$, that is, if $\frac{1}{2}\log 2 \geq (1/\sqrt{2})2^{-(3/2)\log 2}$. The latter can easily be verified. As a result, we obtain from (3.2) that $\mu^+(A) \geq (1/6r)p$. In the same way, for $\frac{1}{2} \leq p < 1$, we have $\mu^+(A) \geq (1/6r)(1-p)$. As a result, we arrive at

$$\mu^+(A) \geq \frac{1}{6r}\min\{\mu(A), 1 - \mu(A)\}. \tag{3.3}$$

To complete the proof, it remains to note that, by Chebyshev's inequality, we have an estimate $\mu\{x \in \mathbf{R}^n\colon |x| \ge \sqrt{3}\,\|\,|x|\,\|_2\} \le \frac{1}{3}$, so that $r \le \sqrt{3}\,\|\,|x|\,\|_2$, and

$$\mu^+(A) \ge \frac{1}{6\sqrt{3}\,\|\,|x|\,\|_2} \min\{\mu(A), 1 - \mu(A)\}. \tag{3.4}$$

Thus, inequality (1.7) holds with $K = 6\sqrt{3}$. □

REMARK. 3.1. In [22], extending and using the "localization lemma" of [28], Kannan, Lovász and Simonovits proved that, for every measurable set A of $\mathbf{R}^n$,

$$\mu^+(A) \ge \frac{\log 2}{\|\,|x|\,\|_1} \mu(A)(1 - \mu(A)). \tag{3.5}$$

Up to numerical constants, inequalities (3.3)–(3.5) are equivalent to each other. The advantage of (3.5) is, however, the property that the constant $\log 2$ is optimal. When μ is uniform distribution on a convex set B with barycenter 0, the quantities r, $\|\,|x|\,\|_2$ and $\|\,|x|\,\|_1$ in (3.3)–(3.5) may be viewed as an "average" diameter of B. It can be essentially smaller in comparison with the usual diameter. For example, it is the case for the regular symplex in $\mathbf{R}^n$, as noted in [22].

PROOF OF THEOREM 1.3. The property (c) is weaker than (a) and (b), so we may assume that

$$d = \|\,|x|\,\|_{L^{\psi}(\mu)} = \inf\left\{t > 0\colon \int \exp\frac{|x|^2}{t^2}\, d\mu(x) \le 2\right\} < +\infty.$$

First we show that, for all measurable $A \subset \mathbf{R}^n$,

$$\mu^+(A) \ge \frac{1}{2d\sqrt{6}} I(\mu(A)), \tag{3.6}$$

where $I = \varphi(\Phi^{-1}(p))$ is the Gaussian isoperimetric function. As in the previous proof, we use (2.4): for all $r > 0$,

$$\mu^+(A) \ge \frac{p\log(1/p) + (1-p)\log(1/(1-p)) + \log \mu(B_r)}{2r}, \qquad p = \mu(A),$$

where B_r is the Euclidean ball with center 0 and radius r. By Chebyshev's inequality,

$$\begin{aligned} 1 - \mu(B_r) &= \mu\{x \in \mathbf{R}^n\colon |x| \ge r\} \le \exp(-r^2/d^2) \int \exp(|x|^2/d^2)\, d\mu(x) \\ &= 2\exp(-r^2/d^2). \end{aligned}$$

Hence, as soon as $2\exp(-r^2/d^2) < 1$,

$$\mu^+(A) \geq \frac{p\log(1/p) + (1-p)\log(1/(1-p)) + \log\big(1 - 2\exp(-r^2/d^2)\big)}{2r}. \tag{3.7}$$

Assume $0 < p \leq \frac{1}{2}$ and apply (3.7) to $r = d\sqrt{\alpha\log(1/p)}$ with α to be chosen later. We have $2\exp(-r^2/d^2) = 2p^\alpha$, and (3.7) becomes

$$\mu^+(A) \geq \frac{1}{2d\sqrt{\alpha}}p\sqrt{\log\frac{1}{p}} + \frac{(1-p)\log(1/(1-p)) + \log(1-2p^\alpha)}{2r}. \tag{3.8}$$

For $\alpha > 1$, the function $v(p) = (1-p)\log(1/(1-p)) + \log(1-2p^\alpha)$ is clearly concave on $[0, \frac{1}{2}]$, and also the requirement $2p^\alpha < 1$ is satisfied. In addition, $v(0) = 0$, so, v will be nonnegative, if $v(\frac{1}{2}) \geq 0$, that is, if $2^{\alpha-2} \geq 1/(2-\sqrt{2})$. In particular, we may take $\alpha = 3$, and from (3.8) we get

$$\mu^+(A) \geq \frac{1}{2d\sqrt{3}}p\sqrt{\log\frac{1}{p}}\,.$$

Now, $p\sqrt{\log(1/p)} \geq (1/\sqrt{2})I(p)$ (cf. the discussion after Theorem 2.2 in [9]), and we arrive at $\mu^+(A) \geq (1/2d\sqrt{6})I(p)$. By a similar argument and noting that $I(1-p) = I(p)$, this inequality holds for all $p \in [\frac{1}{2}, 1)$. As a result, (3.6) has been proved.

Thus, we have shown that (c) $\Rightarrow$ (a). That (a) $\Rightarrow$ (b), as mentioned, was proved by Ledoux [24]: $\rho \geq Kc^2$, for some universal K. This is true in the general situation of an arbitrary probability metric space (M, d, μ). The fact that the optimal constant here is $K = 1$ has become known recently, and we will just mention an argument which yields this result. Let us consider for simplicity the case of a probability measure μ on the Euclidean space $M = \mathbf{R}^n$.

LEMMA 3.2. *Assume that, for all measurable subsets A of $\mathbf{R}^n$,*

$$\mu^+(A) \geq cI(\mu(A)), \tag{3.9}$$

where $c > 0$, and I is the Gaussian isoperimetric function. Then, for all locally Lipschitz functions f on $\mathbf{R}^n$ with values in $[0,1]$,

$$c\left[I\left(\int f\,d\mu\right) - \int I(f)\,d\mu\right] \leq \int |\nabla f|\,d\mu. \tag{3.10}$$

LEMMA 3.3. *Inequality (3.10) implies that, for every locally Lipschitz function f on $\mathbf{R}^n$,*

$$\frac{c^2}{2}\mathrm{Ent}_\mu(f^2) \leq \int |\nabla f|^2\,d\mu. \tag{3.11}$$

The inequality (3.10) represents a functional form for (3.9). It was proved in [6] in the case of the Gaussian measure $\mu = \gamma_n$ with $c = 1$, but the proof extends to arbitrary measures. A more general case involving functions I associated to perfect multiplicative moduli was considered in [7] (Proposition 4.1 and Remark 4.2; cf. also [8] for a full account on this question).

Lemma 3.3 was established, on the basis of Gross's logarithmic inequality, by Bakry and Ledoux in a slightly different context (cf. [2], Theorem 3.2 and the following remark).

We can now combine these two lemmas. Since (3.9) implies (3.11), and since we have proved (3.6), that is, (3.9) with $c = 1/(2d\sqrt{6})$, we arrive at (3.11), that is, at (1.7) with universal constant $K = (2\sqrt{6})^2 = 24$. The proof of Theorem 1.3 is now complete. □

4. Log-concave measures on the real line. In this section we obtain two-sided estimates for the isoperimetric constant $Is(\mu)$, the spectral gap λ_1 and the logarithmic Sobolev constant ρ in the case of an arbitrary log-concave probability measure μ on the real line $\mathbf{R}$. We will consider the identity function $\xi(x) = x$ as a random variable with respect to μ. Denote by ν the measure on $\mathbf{R}$ with density $d\nu(x)/dx = \frac{1}{2}\exp(-|x|)$.

PROPOSITION 4.1. *For every log-concave probability measure μ on $\mathbf{R}$, we have*

$$\frac{1}{3\,\mathrm{Var}(\xi)} \le Is^2(\mu) \le \frac{2}{\mathrm{Var}(\xi)}. \tag{4.1}$$

Equality on the left is possible if and only if μ is a uniform distribution on some finite interval, while equality on the right is possible if and only if μ is image of ν under an affine transformation.

As will be explained in the proof of this proposition,

$$Is(\mu) = 2f(m), \tag{4.2}$$

where f and m are, respectively, the density and the median of μ. When μ is symmetric around 0, $m = 0$ and (4.1) becomes

$$\tfrac{1}{3} \le 8f(0)^2 \int_0^{+\infty} x^2 f(x)\,dx \le 2. \tag{4.3}$$

The inequality on the right, for decreasing log-concave functions f on $[0, +\infty)$, has been established by Ball [4] in his study of sections of convex isotropic bodies in Euclidean space [the left inequality in (4.3) is a particular case of a more general observation of Hensley]. Let us recall that a convex compact set $B \subset \mathbf{R}^n$ of Lebesgue measure $\mathrm{Vol}_n(B) > 0$ is called isotropic if:

1. B is symmetric around the origin.
2. The correlation operator of the normalized Lebesgue measure λ_B on B is up to a constant the identity operator.

In other words, the property (b) requires that, with respect to λ_B, the variances of linear functionals on $\mathbf{R}^n$ of Euclidean norm one are equal to each other.

Let B be isotropic with (for definiteness) $\mathrm{Vol}_n(B) = 1$. Distributions of linear functions with respect to λ_B are log-concave on $\mathbf{R}$ as linear images of a log-concave measure. If we take two arbitrary linear functions g_i $(i = 1, 2)$ of Euclidean norm 1 and denote by f_i the densities of their distributions, then clearly $f_i(0) = \mathrm{Vol}_{n-1}(B \cap H_i)$ for $H_i = \{x \in \mathbf{R}^n\colon\ g_i(x) = 0\}$, while $\int x^2 f_1(x)\,dx = \int x^2 f_2(x)\,dx$ due to (1) and (2). Using (4.3), we then can conclude that for all hyperplanes H_1 and H_2 in $\mathbf{R}^n$ containing the origin,

$$\frac{\mathrm{Vol}_{n-1}(B \cap H_1)}{\mathrm{Vol}_{n-1}(B \cap H_2)} \le \sqrt{6}\,. \tag{4.4}$$

This is an argument which led Ball from (4.3) to (4.4) (he actually considered intersections of B with k-codimensional subspaces). The constant $\sqrt{6}$ can be improved for fixed values n; however, it is optimal if the dimension is arbitrary. With a suboptimal constant, this inequality was discovered by Hensley [21] and later was rediscovered by Milman (according to [15]; cf. also [18] for related results). In view of (4.1) and (4.2), with the same argument we have the following generalization.

COROLLARY 4.2. *Let B be a convex compact set in $\mathbf{R}^n$ of positive Lebesgue measure which satisfies* (2). *Then, for all hyperplanes H_1 and H_2 in $\mathbf{R}^n$ which divide B into pieces of equal volumes, the inequality* (4.4) *still holds.*

Thus, we can drop the symmetry assumption (1) and involve in (4.4) many "nonsymmetric" sets such as the regular simplex in $\mathbf{R}^n$ (like a tetrahedron). Let us also observe that the assumption (2), that is, the property

$$\langle R_B g_1, g_2 \rangle \equiv \int_B \langle g_1, x - x_0 \rangle \langle g_2, x - x_0 \rangle\, dx = c_B \langle g_1, g_2 \rangle \qquad \text{for all } g_1, g_2 \in \mathbf{R}^n$$

(where x_0 is barycenter of B), has a matter of normalization, only. One can start, more generally, from an arbitrary convex set B and consider two hyperplanes $H_i = \{x \in \mathbf{R}^n\colon\ g_i(x) = c_i\}$ which divide B into equal pieces ($c_i \in \mathbf{R}$, and g_i are linear functionals of Euclidean norm 1). Then (4.4) should turn into

$$\frac{\mathrm{Vol}_{n-1}(B \cap H_1)}{\mathrm{Vol}_{n-1}(B \cap H_2)} \le \sqrt{6\,\frac{\mathrm{Var}(g_2)}{\mathrm{Var}(g_1)}} \le \sqrt{6\,\frac{\alpha_n}{\alpha_1}}\,,$$

where the variances of g_i are with respect to λ_B, and α_1 and α_n are the smallest and the largest eigenvalues of the correlation operator R_B of λ_B.

Using the Cheeger theorem, we can now derive from (4.1) some estimates for λ_1.

COROLLARY 4.3. *For every log-concave probability measure* μ *on* $\mathbf{R}$, *we have*

$$\frac{1}{12\,\mathrm{Var}(\xi)} \le \lambda_1 \le \frac{1}{\mathrm{Var}(\xi)}.$$

We do not know, however, if the constant on the left is optimal. The right-hand side inequality trivially holds for all μ, without log-concavity assumption. An equality here is possible only for Gaussian measures; such a characterization was established by Borovkov and Utev [14] who studied a connection of Poincaré-type inequalities with the central limit theorem.

At last we have similar estimates for the logarithmic Sobolev constant.

PROPOSITION 4.4. *For every log-concave probability measure* μ *on* $\mathbf{R}$, *we have*

$$\frac{1}{24\|\xi - \mathbf{E}\xi\|_\psi^2} \le \rho \le \frac{8}{3\|\xi - \mathbf{E}\xi\|_\psi^2}, \tag{4.5}$$

where $\|\cdot\|_\psi$ *is the Orlicz norm associated with the function* $\psi(x) = \exp(x^2) - 1$.

Here, the constant on the left is apparently far from the best. The constant $8/3$ on the right is however optimal (equality is achieved for Gaussian measures). As above, the right-hand side inequality holds for all μ, without log-concavity assumption (cf. [1]).

PROOF OF PROPOSITION 4.1. Let (a, b) be a minimal interval, finite or not, which supports μ. The distribution function $F(x) = \mu((-\infty, x])$ is continuously differentiable and is increasing on (a, b), and, moreover, the density of μ, $f = F'$, is logarithmically concave on that interval. Up to a shift parameter, there is 1−1 correspondence between the family of all log-concave probability measures μ on $\mathbf{R}$ and the family of all concave positive functions I_μ on $(0, 1)$, defined by

$$I_\mu(p) = f(F^{-1}(p)), \qquad 0 < p < 1,$$

where $F^{-1}\colon (0,1) \to (a, b)$ is the inverse of F (cf. [5], Proposition A.1). For example, the function

$$I_\nu(p) = \min\{p, 1-p\}, \qquad 0 < p < 1,$$

determines the two-sided exponential measure ν. Thus, we can pass from I_μ to μ using an identity

$$F^{-1}(p) - F^{-1}(q) = \int_q^p \frac{dt}{I_\mu(t)}, \qquad p, q \in (0,1), \tag{4.6}$$

and may think of μ as of an arbitrary concave positive function I_μ on (0, 1). By concavity, for all $p \in (0, 1)$,

$$I_\mu(p) \geq 2I_\mu(\tfrac{1}{2})\min\{p, 1-p\} = 2I_\mu(\tfrac{1}{2})I_\nu(p), \tag{4.7}$$

since there is equality at $p = \frac{1}{2}$. Therefore,

$$\inf_{0<p<1} \frac{I_\mu(p)}{\min\{p, 1-p\}} = 2I_\mu(\tfrac{1}{2}).$$

On the other hand, by Theorem 1.3 from [11], there is a general identity,

$$Is(\mu) = \inf_{a<x<b} \frac{f(x)}{\min\{F(x), 1-F(x)\}} = \inf_{0<p<1} \frac{I_\mu(p)}{\min\{p, 1-p\}}.$$

Since, by definition, $I_\mu(\frac{1}{2}) = f(m)$, we get $Is(\mu) = 2f(m)$, that is, (4.2).

Now we will try to estimate $\mathrm{Var}(\xi)$ using the fact that the function F^{-1} has distribution μ with respect to Lebesgue measure on (0, 1). In view of (4.6) and (4.7),

$$\begin{aligned} \mathrm{Var}(\xi) &= \frac{1}{2}\int_0^1\int_0^1 \left(F^{-1}(p) - F^{-1}(q)\right)^2 dp\,dq \\ &= \frac{1}{2}\int_0^1\int_0^1 \left(\int_q^p \frac{dt}{I_\mu(t)}\right)^2 dp\,dq \\ &\leq \frac{1}{2}\int_0^1\int_0^1 \left(\int_q^p \frac{dt}{2I_\mu(\frac{1}{2})I_\nu(t)}\right)^2 dp\,dq. \end{aligned} \tag{4.8}$$

By the same reason, if a random variable η has distribution ν,

$$\mathrm{Var}(\eta) = \frac{1}{2}\int_0^1\int_0^1 \left(\int_q^p \frac{dt}{I_\nu(t)}\right)^2 dp\,dq.$$

Since $\mathrm{Var}(\eta) = 2$ and $Is(\mu) = 2I_\mu(\frac{1}{2})$, we can conclude that $\mathrm{Var}(\xi) \leq 2/Is^2(\mu)$. This corresponds to the right inequality in (4.1). Also note that equality in (4.8) is possible only if, for all $t \in (0, 1)$, $I_\mu(t) = 2I_\mu(\frac{1}{2})I_\nu(t)$, that is, if μ is an image of ν under an affine function.

To prove the left-hand side inequality in (4.1), we should estimate the function I_μ from above. To this aim, let us again note that, by concavity, for some θ,

$$I_\mu(p) \leq I_\theta(p) \equiv I_\mu(\tfrac{1}{2}) + \theta(p - \tfrac{1}{2}), \qquad 0 < p < 1.$$

Because I_μ is nonnegative, we have a restriction $|\theta| \leq 2I_\mu(\frac{1}{2})$. As in (4.8),

$$\begin{aligned} \mathrm{Var}(\xi) &= \frac{1}{2}\int_0^1\int_0^1 \left(\int_q^p \frac{dt}{I_\mu(t)}\right)^2 dp\,dq \\ &\geq \frac{1}{2}\int_0^1\int_0^1 \left(\int_q^p \frac{dt}{I_\theta(t)}\right)^2 dp\,dq \equiv u(\theta). \end{aligned}$$

The function $\theta \to 1/I_\theta(t)$ is convex, so is the function u. Since the later is also symmetric around 0, we get $u(\theta) \geq u(0)$, for all θ. Therefore,

$$\begin{aligned}\operatorname{Var}(\xi) &\geq \frac{1}{2}\int_0^1\int_0^1\left(\int_q^p \frac{dt}{I_0(t)}\right)^2 dp\,dq\\ &= \frac{1}{2I_\mu^2(\frac{1}{2})}\int_0^1\int_0^1 (p-q)^2\,dp\,dq = \frac{1}{3Is^2(\mu)},\end{aligned}$$

where we have used once more the identity (4.2). Thus, we arrived at the left-hand side inequality in (4.1). It should also be clear that an equality is possible only if $I_\mu(t) = I_0(t) = I_\mu(\frac{1}{2})$, for all $t \in (0,1)$. But this is equivalent to saying that μ is a uniform distribution on some interval of length $1/I_\mu(\frac{1}{2})$. Proposition 4.1 follows. □

PROOF OF PROPOSITION 4.4. We may assume that $x_0 = \mathbf{E}\xi = 0$ and that $d = \|\xi\|_\psi$ is finite. The left inequality in (4.5) has been proved in the multidimensional case with the same constant. To prove the converse estimate, one can use the following inequality:

$$\int \exp(tf^2)\,d\mu \leq \frac{1}{\sqrt{1-2t/\rho}}, \qquad 0 \leq t < \rho/2, \tag{4.9}$$

which holds for all f with $\|f\|_{\mathrm{Lip}} \leq 1$ and $\int f\,d\mu = 0$. This is an observation of Gross [1] based on Ledoux's inequality for exponential moments of f ([25], [26]). We come to the right-hand side inequality in (4.5), applying (4.9) to $f = \xi$ and $t = 3\rho/8$: in this case $\int \exp(t\xi^2)\,d\mu \leq 2$, hence $\|\xi\|_\psi \leq 1/\sqrt{t} = \sqrt{8/3\rho}$.

5. Norms of random vectors with log-concave distribution. Let X be a random vector in a linear normed space $(E, \|\cdot\|)$ with a log-concave distribution μ. By definition, μ is log-concave, if for all nonempty μ-measurable subsets A and B of E and for all $\lambda \in (0,1)$,

$$\mu_*((1-\lambda)A + \lambda B) \geq \mu(A)^{1-\lambda}\mu(B)^\lambda, \tag{5.1}$$

where μ_* is the inner measure [for a possible case when $(1-\lambda)A + \lambda B$ is not measurable]. For E finite dimensional and μ absolutely continuous, this definition reduces, by Prékopa-Leindler's theorem, to log-concavity of density of μ, that is, to the original definition (cf. also [12]). We will assume that μ is not a Dirac measure at a point.

Distribution μ_X of $\|X\|$ is already not a log-concave measure (on $\mathbf{R}$), but it inherits some of the properties of log-concave measures which we studied before. In particular, we have the following proposition.

PROPOSITION 5.1. *For any random vector X in a linear normed space $(E, \|\cdot\|)$ with a log-concave distribution μ,*

$$Is(\mu_X) \geq \frac{1}{6r}, \tag{5.2}$$

where r is a quantile of $\|X\|$ of order $2/3$.

This is exactly inequality (3.3) which was obtained in the proof of Theorem 1.2 for a log-concave measure μ on $\mathbf{R}^n$. As we will see, that proof can easily be adapted to the case of the norm-image of μ. However, we do not know if the (much stronger) left inequality of Proposition 4.1, $Is^2(\mu_X) \geq 1/K \operatorname{Var}(\|X\|)$, remains true for such measures μ_X.

Inequality (5.2), as noted before, is equivalent to the property that the nondecreasing map T: $\mathbf{R} \to \mathbf{R}$ which transforms the two-sided exponential measure ν into μ_X has Lipschitz norm $\|T\|_{\mathrm{Lip}} \leq 6r$. Here, as well as in (3.3), the quantile r of order $2/3$ can actually be replaced by median and by other quantiles r_p at the expense of the constant in front of r_p. This cannot, however, be deduced from the Borell Lemma 3.1 in its present form because of the assumption $\theta > 1/2$. An attempt to recover an exponential decay, as in Lemma 3.1, for $1 - \mu(tB)$ with an arbitrary fixed value of $\mu(B)$ leads to the problem of small probabilities,

$$F(t) = \mu\{x \in E: \|x\| \leq t\} = \mu(tB), \qquad B = \{x \in E: \|x\| \leq 1\}$$

(with t small). In connection with Khinchine–Kahane inequalities, the problem has been studied by Latala [23] who showed that there exist constants C_p depending on $p \in (0,1)$ only, such that

$$F(t) \leq C_p t \quad \text{if } \mu(B) \leq p, 0 \leq t \leq 1. \tag{5.3}$$

It then follows immediately that the quantile r of Proposition 5.1 can be estimated in terms of quantiles r_p of order $p \in (0, 2/3]$ as $r \geq r_p C_p/p$. Inequality (5.3) has recently been quantified by Gideon [19] showing that

$$F(t) \leq 2t \log \frac{1}{1 - \mu(B)}, \qquad 0 \leq t \leq 1.$$

This was derived from a sharpened form of Borell's Lemma 3.1, namely,

$$1 - \mu(tB) \leq (1 - \mu(B))^{(t+1)/2}, \qquad t \geq 1 \tag{5.4}$$

[now without the condition $\theta = \mu(B) > 1/2$]. For B the Euclidean ball in $\mathbf{R}^n$, the above inequality was earlier obtained by Lovász and Simonovits as a consequence of a "localization" lemma [28], and Gideon adapted their proof to get the general result. Clearly, inequality (5.4) may also be used to improve numerical constants in Theorems 1.2 and 1.3, but we have chosen the root independent of the localization method. Here we would like to show another way giving a property somewhat related to (5.3).

PROPOSITION 5.2. *In the interval* $0 < F(t) \le e/(e+1)$, *the function* $F(t)/t^{1/(2e)}$ *is nondecreasing.*

PROOF. By definition (5.1), the function $\log F$ is concave in the interval $\Delta = \{t\colon 0 < F(t) < 1\}$, so F has a left-continuous Radon–Nikodym derivative f on Δ. Moreover, using a simple approximation argument, we may assume that the function f is continuous on Δ. For $t, s \in \Delta$, $s > t$, consider a set $A = \{x \in E\colon t < \|x\| \le s\}$. Since for $\lambda \in (0, \frac{1}{2}]$,

$$(1-\lambda)A + \lambda(tB) \subset \{x \in E\colon (1-2\lambda)t < \|x\| \le (1-\lambda)s + \lambda t\},$$

we obtain, by (5.1), that

$$F((1-\lambda)s + \lambda t) - F((1-2\lambda)t) \ge (F(s) - F(t))^{1-\lambda} F(t)^{\lambda}. \tag{5.5}$$

There is equality in (5.5) for $\lambda = 0$, and comparing the derivatives of both sides at $\lambda = 0$, we arrive at

$$f(s)(t-s) + 2f(t)t \ge (F(s) - F(t))\log\frac{F(t)}{F(s) - F(t)}.$$

Hence, since $t < s$,

$$2f(t)t \ge (F(s) - F(t))\log\frac{F(t)}{F(s) - F(t)}. \tag{5.6}$$

Now, given $t \in \Delta$ such that $F(t) < e/(e+1)$, we can choose s so that $F(s) - F(t) = F(t)/e$, and then (5.6) becomes

$$2f(t)t \ge \frac{F(t)}{e},$$

that is, $f(t)/F(t) \ge 1/(2et)$. However, this is equivalent to saying that the function $\log F(t) - \log t^{1/(2e)}$ is nondecreasing. Proposition 5.2 follows. $\square$

The example $(E, \|\cdot\|, \mu) = (\mathbf{R}, |\cdot|, \nu)$ shows that the exponent $1/(2e)$ in Proposition 5.2 cannot be replaced by 1 even for smaller intervals. However, in the present form the above property is, for example, sufficient to yield together with Lemma 3.1 an inequality for arithmetic and geometric means,

$$\|X\|_\alpha \equiv (\mathbf{E}\|X\|^\alpha)^{1/\alpha} \le c(\alpha)\|X\|_0,$$

where $\|X\|_0 \equiv \lim_{\alpha\to 0}\|X\|_\alpha = \exp \mathbf{E}\log\|X\|$, and where the constants $c(\alpha)$ depend on $\alpha > 0$ only. This was proved in [23] on the basis of (5.3).

PROOF OF PROPOSITION 5.1. Set $\Delta = \{t\colon 0 < F(t) < 1\}$, $r_0 = \inf\Delta$ (note that μ_X may have an atom at r_0). By log-concavity of F, for all $s \in \Delta$, $s > r_0$, every $t \ge 1$ and every $\lambda \in (0,1)$,

$$F((1-\lambda)s + \lambda tr) \ge F(s)^{1-\lambda} F(tr)^{\lambda}. \tag{5.7}$$

Applying now (5.1) to $A = \{x \in E: \|x\| > s\}$ and to rtB with sufficiently small λ so that $(1-\lambda)A + \lambda(rtB) \subset \{x \in E: \|x\| > (1-\lambda)s - \lambda rt > r_0\}$, we obtain for such λ that

$$1 - F((1-\lambda)s - \lambda tr) \geq (1 - F(s))^{1-\lambda} F(tr)^{\lambda}. \tag{5.8}$$

Then (5.7) and (5.8) become equalities for $\lambda = 0$, and comparing the corresponding derivatives of both sides at $\lambda = 0$, we arrive, respectively, at

$$f(s)(tr - s) \geq F(s)\log F(tr) - F(s)\log F(s),$$
$$f(s)(tr + s) \geq (1 - F(s))\log F(tr) - (1 - F(s))\log(1 - F(s)),$$

where f is the lower derivative of F. Setting $F(s) = p$, $B_r = rB$, and summing the above inequalities, we get

$$f(s) \geq \frac{p\log(1/p) + (1-p)\log(1/(1-p)) + \log \mu(tB_r)}{2rt}. \tag{5.9}$$

This is exactly the first inequality in (3.2). Optimizing the right-hand side of (5.9) over $t \geq 1$ as in the pass from (3.2) to (3.3), we come to

$$f(s) \geq \frac{1}{6r}\min\{F(s), 1 - F(s)\},$$

which holds at least for almost all $s \in \mathbf{R}$ (with respect to Lebesgue measure). It remains to recall that, in general,

$$Is(\mu) = \operatorname{essinf}_s \frac{f(s)}{\min\{F(s), 1 - F(s)\}}$$

(cf. [11], Theorem 1.3). Proposition 5.1 is thus proved. □

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DEPARTMENT OF MATHEMATICS
SYKTYVKAR UNIVERSITY
167001 SYKTYVKAR
RUSSIA