THEORY PROBAB. APPL.
Vol. 45, No. 4

Translated from Russian Journal

SOME GENERALIZATIONS OF PROKHOROV'S RESULTS ON KHINCHIN-TYPE INEQUALITIES FOR POLYNOMIALS*

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Abstract. In this paper, we prove the Khinchin-type inequality $(\mathbf{E}|f(\xi)|^p)^{1/p} \leqq C(p,d)\,\mathbf{E}|f(\xi)|$ for polynomials in components of random vectors in a space with logarithmically concave density.

Key words. Khinchin-type inequalities

PII. S0040585X97978580

Some years ago Prokhorov investigated the Khinchin-type inequalities

$$\left(\mathbf{E}|f(\xi)|^2\right)^{1/2} \leqq C(d)\,\mathbf{E}|f(\xi)| \tag{1}$$

in the class of polynomials f of degree n. In [9], using the expansion in Hermite polynomials he proved this inequality in the case of normally distributed random variables ξ. In [10] the Laguerre polynomials were used to prove inequalities of the form (1) for Gamma-distributed random variables. Like the results themselves, of indisputable interest is the approach to (1) based on the application of the well-known Markov inequality connecting the maximum values of a polynomial and its derivative on a finite interval. In both cases this approach made it possible to obtain inequality (1) with regular exponential growth of the coefficients $C(d)$ as the power d increases. The following statement proved in [2] is a natural generalization of the results mentioned above.

THEOREM 1. *Inequality* (1) *holds in the class of all polynomials f on $\mathbf{R}$ of degree d in arbitrary sets of random variables $\xi = (\xi_1, \dots, \xi_n)$ having joint logarithmically concave density on $\mathbf{R}^n$, with the coefficient of the form $C(d) = C^d$, where C is a universal constant.*

Recall that a nonnegative function ρ in $\mathbf{R}^n$ is called logarithmically concave if it satisfies the inequality $\rho((1-t)\,x + ty) \geqq \rho(x)^{1-t}\rho(y)^t$ for all $x, y \in \mathbf{R}^n$ and $t \in (0,1)$.

In the one-dimensional case ($n = 1$) Theorem 1 can be easily proved if the rate of increase of coefficients $C(d)$ is not taken into account (see [3]). However, in the form presented above the statement becomes much more delicate. For the first time the question of comparability of the L^p- and L^1-norms for multivariate polynomials, with respect to the uniform distribution λ_K on a convex compact body K in $\mathbf{R}^n$, was formulated by Milman. The solution of this problem using the Knothe parametrization was given by Bourgain [4] who obtained an estimate for the probabilities of large deviations $\lambda_K\{|f| \geqq t\mathbf{E}|f|\} \leqq \exp\{-t^{c/d}\}$, $t \geqq t_0$, where $c \in (0,1)$ and $t_0 > 0$ are absolute constants. Equivalently, for a random vector ξ with the uniform distribution on K, the Khinchin-type inequality

$$\left(\mathbf{E}|f(\xi)|^p\right)^{1/p} \leqq C(p,d)\,\mathbf{E}|f(\xi)| \tag{2}$$

holds with constants of the form $C(p,d) = (Cpd)^{Cd}$, $C > 1$. Previously, Gromov and Milman [5] considered the linear case $d = 1$. From a special case of Bourgain's result (as applied to the uniform distribution on the cube $[-1,1]^n$ with growing dimension n), by virtue of the central limit theorem, inequality (1) follows for normally distributed random

*Received by the editors May 24, 2000. This work was supported by Russian Foundation for Basic Research grant 99-01-00247.

http://www.siam.org/journals/tvp/45-4/97858.html

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variables ξ, however, with the worse increase rate of the constants $C(d)$ as compared with Prokhorov's result. Later Lovász and Simonovits [8] introduced the so-called localization method which makes it possible to reduce some relations between multidimensional integrals to those between one-dimensional integrals (in connection with isoperimetric problems on a sphere, ideas were earlier developed in [6]; also see [1]). Developing this method, Kannan, Lovász, and Simonovits, in particular, proved an important theorem [7, Theorem 2.7] from which we obtain the following result.

THEOREM 2. *Let $p \geqq 1$, $C \geqq 1$, and let f be a continuous function on $\mathbf{R}^n$. The inequality $(\mathbf{E}|f(\xi)|^p)^{1/p} \leqq C\mathbf{E}|f(\xi)|$ holds for all random vectors ξ having logarithmically concave density on $\mathbf{R}^n$ if and only if it holds for random vectors of the form $\xi = a+\eta b$, where η is a $[0,1]$-valued random variable with the density $\lambda\, e^{-\lambda x}/(1-e^{-\lambda})$, $x \in [0,1]$ ($a, b \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$ are arbitrary parameters).*

If $f = f(x_1, \dots, x_n)$ is a polynomial of degree d in n variables and $t \in \mathbf{R}$, then $f(a+tb)$ is a polynomial of degree no higher than d in one variable. Therefore, Theorem 2 reduces inequalities (1) and (2) to the one-dimensional case $n = 1$. Moreover, to estimate the constant coefficients $C(p,d)$, choosing an appropriate scaling if necessary, one can assume that the random variable ξ has the truncated exponential distribution ν_n with the density

$$\frac{d\nu_u(x)}{dx} = \frac{e^{-x}}{1-e^{-u}}\, 1_{(0,u)}(x), \quad x \in \mathbf{R},\ u > 0$$

(as the limit case, we obtain the standard exponential distribution $\nu_{+\infty} = \nu$ with the density e^{-x}, $x > 0$).

Thus, Theorem 2 considerably simplifies the situation. However, the question of the extremal polynomials and distributions providing the values of optimal constants in (2) still remains open and some additional means should be brought in to investigate the asymptotic behavior of $C(p,d)$ as a function of two parameters. As was shown in [2], combining the approach of Prokhorov with Theorem 2 one can obtain Theorem 1 (with $C = e^{11}$). If inequality (1) is sequentially applied to the polynomials $f^2, f^4, \dots, f^{2^k}$, then with $p \geqq 2$ we arrive at inequality (2) with coefficients $C(p,d) = p^{Cd}$, where $C > 1$ is a constant. In this paper, we will demonstrate a way to obtain a more accurate assertion.

THEOREM 3. *Inequality (2) holds in the class of all polynomials f on $\mathbf{R}^n$ of degree d in arbitrary random vectors ξ with logarithmically concave density on $\mathbf{R}^n$; moreover, the coefficient has the form $C(p,d) = (Cp)^d$, where C is a universal constant.*

Note that, as is demonstrated by the example of the polynomial $f(x) = x^d$ and a random variable having the standard exponential distribution ν, the optimal constant in (2) must satisfy the inequality $C(p,d) \geqq cp^d/d^{(p-1)/(2p)}$ with some $c > 0$.

To prove Theorem 3 we need some preparations. We rely upon Theorems 1 and 2 and use the reasoning from [10].

LEMMA. *Let the random variable ξ have the distribution ν. For any polynomial f on $\mathbf{R}$ of degree $d \geqq 1$ and any $p \geqq 1$ we have*

$$\left(\mathbf{E}|f(\xi)|^p\right)^{1/p} \leqq (e^{12}p)^d\, \mathbf{E}|f(\xi)|.$$

Proof. Denote $\|\eta\|_p = (\mathbf{E}|\eta|^p)^{1/p}$. By Theorem 1 with $C = e^{11}$, we can assume that $p \geqq 2$. By virtue of homogeneity, we can assume that $\|f(\xi)\|_2^2 = \int_0^{+\infty} x^2|f(x)|^2 e^{-x}dx = 1$. Introduce the Laguerre polynomials

$$L_k(x) = \frac{e^x}{k!}\,\frac{d^k}{dx^k}\,(x^k e^{-x}) = \sum_{j=0}^{k} (-1)^j C_k^j\, \frac{x^j}{j!}, \quad k = 0, 1, \dots . \tag{3}$$

They constitute a complete orthonormal system in $L^2(\nu)$; furthermore, f is representable as $f = \sum_{k=0}^d a_k L_k$, where $\sum_{k=0}^d |a_k|^2 = 1$. Therefore, $|f|^2 \leqq \sum_{k=0}^d |L_k|^2$ and

$$\|f(\xi)\|_p^2 = \big\||f(\xi)|^2\big\|_{p/2} \leqq \sum_{k=0}^{d} \big\|\,|L_k(\xi)|^2\big\|_{p/2} = \sum_{k=0}^{d} \|L_k(\xi)\|_p^2. \tag{4}$$

It can be easily deduced from the Stirling formula that $\Gamma(pj+1)^{1/p} \leqq p^j\Gamma(j+1)$ for all $p \geqq 2$ and all integers $j \geqq 0$. By (3) we obtain

$$\|L_k(\xi)\|_p \leqq \sum_{j=0}^{k} C_k^j \frac{\|\xi^j\|_p}{j!} = \sum_{j=0}^{k} C_k^j \frac{\Gamma(pj+1)^{1/p}}{j!} \leqq \sum_{j=0}^{k} C_k^j p^j = (p+1)^k.$$

Applying (4), we conclude that

$$\|f(\xi)\|_p^2 \leqq \sum_{k=0}^{d} (p+1)^{2k} \leqq (d+1)(p+1)^{2d} < (2p)^{2d}, \quad d \geqq 2,$$

and $\|f(\xi)\|_p^2 \leqq 1+(p+1)^2 < (2p)^2$ for $d=1$. Thus, $\|f(\xi)\|_p \leqq (2p)^d\|f(\xi)\|_2$. Now it remains to apply Theorem 1. The lemma is proved.

Proof of Theorem 3. By virtue of Theorem 2 it suffices to prove the inequality

$$\big(\mathbf{E}|f(\xi_u)|^p\big)^{1/p} \leqq (Cp)^d\, \mathbf{E}|f(\xi_u)|, \quad p \geqq 1,\ u > 0, \tag{5}$$

with some universal constant C, where f is an arbitrary polynomial of degree $d \geqq 1$, and ξ_u is a random variable with distribution ν_u. As above, $\|\eta\|_p = (\mathbf{E}|\eta|^p)^{1/p}$ $(1 \leqq p \leqq +\infty)$.

Step 1. $0 < u < 24d$. Let $x_0 \in [0,u]$ be the point of maximum of $|f|$ on the interval $[0,u]$. Without loss of generality, let $f(x_0) > 0$. By the Markov inequality we have

$$\|f'(\xi_u)\|_\infty = \max_{0\leqq x\leqq u} |f'(x)| \leqq \frac{2d^2}{u} \max_{0\leqq x\leqq u} |f(x)| = \frac{2d^2}{u}\, \|f(\xi_u)\|_\infty.$$

Therefore, for all $x \in [0,u]$

$$\begin{aligned} f(x) &\geqq f(x_0) - \|f'(\xi_u)\|_\infty |x-x_0| \\ &\geqq f(x_0) - \frac{2d^2}{u}\, \|f(\xi_u)\|_\infty |x-x_0| = \left(1 - \frac{2d^2}{u}\, |x-x_0|\right) \|f(\xi_u)\|_\infty. \end{aligned}$$

Hence, for points x from the interval $\delta = [x_1,x_2] \equiv [x_0 - u/(4d^2),\ x_0 + u/(4d^2)] \cap [0,u]$ the estimate $f(x) \geqq \frac{1}{2}\|f(\xi_u)\|_\infty$ is valid. Consequently,

$$\|f(\xi_u)\|_1 \geqq \int_\delta f(x)\, d\nu_u(x) \geqq \frac{1}{2}\, \|f(\xi_u)\|_\infty \nu_u(\delta). \tag{6}$$

Moreover, since $x_2 - x_1 \geqq u/(4d^2)$, for some point $x_3 \in [x_1,x_2]$ we have

$$\nu_u(\delta) = \frac{e^{-x_1} - e^{-x_2}}{1-e^{-u}} = \frac{x_2 - x_1}{1-e^{-u}}\, e^{-x_3} \geqq \frac{1}{4d^2}\, \frac{u}{1-e^{-u}}\, e^{-24d} \geqq \frac{1}{4d^2}\, e^{-24d}.$$

Combining this with (6), we obtain $8d^2e^{24d}\|f(\xi_u)\|_1 \geqq \|f(\xi_u)\|_\infty \geqq \|f(\xi_u)\|_p$. Hence (5) easily follows.

Step 2. $u \geqq 24d$. Let ξ be a random variable with distribution ν. Applying the Cauchy–Bunyakovskii inequality and Theorem 1 (again with $C = e^{11}$), we have

$$\int_u^\infty |f(x)|\, e^{-x}\, dx \leqq \|f(\xi)\|_2 \big[\nu(u,+\infty)\big]^{1/2} \leqq e^{11d-u/2}\|f(\xi)\|_1 \leqq e^{-d}\|f(\xi)\|_1.$$

Therefore,

$$\frac{1}{1-e^{-u}} \int_u^\infty |f(x)|\, e^{-x}\, dx \leqq \frac{1}{e(1-e^{-24})}\, \|f(\xi)\|_1 \leqq \frac{1}{2}\|f(\xi)\|_1,$$

which is equivalent to the inequality $\mathbf{E}|f(\xi_u)| \geqq \frac{1}{2}\mathbf{E}|f(\xi)|$. Finally, by virtue of the lemma we have

$$\begin{aligned} \mathbf{E}|f(\xi_u)|^p &= \frac{1}{1-e^{-u}} \int_0^u |f(x)|^p\, e^{-x}\, dx \leqq \frac{1}{1-e^{-24}} \int_0^\infty \|f(x)\|^p e^{-x} dx \\ &= \frac{1}{1-e^{-24}}\, \mathbf{E}\|f(\xi)\|^p \leqq 2(e^{12}p)^{pd}\big(\mathbf{E}|f(\xi)|\big)^p \leqq 2^{p+1}(e^{12}p)^{pd}\big(\mathbf{E}|f(\xi_u)|\big)^p. \end{aligned}$$

So, we arrive at inequality (5) with $C = 4e^{12}$. Theorem 3 is proved.

Acknowledgment. The author would like to express his gratitude to Yu. V. Prokhorov for stimulating discussions.

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