LOCALIZATION PROOF OF THE BAKRY–LEDOUX ISOPERIMETRIC INEQUALITY AND SOME APPLICATIONS*

S. G. BOBKOV†

(*Translated by M. V. Khatuntseva*)

Abstract. A Gaussian-type isoperimetric inequality due to D. Bakry and M. Ledoux is proved by means of the localization lemma of Lovász and Simonovits. Some applications on sharp large deviations are given.

Key words. isoperimetric inequality, localization, large deviations

PII. S0040585X97979688

Let γ_n be a standard Gaussian measure in $\mathbf{R}^n$. Denote by Φ the standard normal distribution function,

$$\Phi(x) = \gamma_1(-\infty, x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt, \qquad x \in \mathbf{R},$$

and by $\Phi^{-1}\colon [0,1] \to [-\infty, +\infty]$ the inverse function. As usual, $\varphi(x) = (2\pi)^{-1/2}\, e^{-x^2/2}$.

The isoperimetric theorem in the Gauss space $(\mathbf{R}^n, \gamma_n)$ states that for all measurable $A \subset \mathbf{R}^n$ and all $h > 0$

$$\gamma_n(A^h) \geqq \Phi\left(\Phi^{-1}(\gamma_n(A)) + h\right), \tag{1}$$

where $A^h = \{x \in \mathbf{R}^n \colon \exists\, a \in A\colon |a - x| < h\}$ is an open h-neighborhood of the set A in a sense of ordinary Euclidean distance in $\mathbf{R}^n$. This important property of the Gaussian measure was discovered by Sudakov and Tsirel'son [26] and Borell [10]; they obtained inequality (1) as a corollary of the Lévy–Schmidt theorem on isoperimetric properties of balls on a sphere. Later Ehrhard [12] proposed a direct proof introducing the Gaussian symmetrization technique (see [9], [11], [22]). As was shown in [5], one can also derive (1) using an integral differential inequality for independent Bernoulli variables. Some abstract generalizations were studied in [6]; see also [3].

We consider the following natural generalization of the Gaussian isoperimetric inequality which was obtained by Bakry and Ledoux with use of a well developed semigroup technique [2], [21]. Let μ be an absolutely continuous probability measure on $\mathbf{R}^n$ with a density of the form $d\mu(x)/dx = e^{-U(x)}$, where U is a convex twice continuously differentiable function, whose second derivative satisfies (in the sense of comparing positively definite matrices in all points of the space) inequality $c\,U''(x) \geqq \mathrm{Id}$ ($c > 0$ is some constant). Then

$$\mu(A^h) \geqq \Phi\left(\Phi^{-1}(\mu(A)) + \frac{h}{c}\right). \tag{2}$$

In frames of the more abstract scheme of diffusion generators, Bakry and Ledoux consider the statement of such type as an infinite dimensional analogue of the Gromov theorem [13], [14] generalizing the isoperimetric inequality on a sphere onto the Riemannian manifolds with the Ricci curve bounded from below. Considering the same measures on $\mathbf{R}^n$ and, without loss

*Received by the editors February 20, 2001. This work was supported by NSF grant DMS-0103929.

http://www.siam.org/journals/tvp/47-2/97968.html

†School of Mathematics, University of Minnesota, Minneapolis, MN 55455 (bobkov@math.umn.edu).

of generality, the case $c = 1$ in inequality (2), we can give the following equivalent statement of the Bakry–Ledoux result.

THEOREM 1. *If a probability measure μ on $\mathbf{R}^n$ has a log-concave density with respect to the measure γ_n, then for all measurable $A \subset \mathbf{R}^n$ and all $h > 0$*

$$\mu(A^h) \geqq \Phi\bigl(\Phi^{-1}(\mu(A)) + h\bigr).$$

Recall that a nonnegative function ρ defined on $\mathbf{R}^n$ is called log-concave if, for all $x, y \in \mathbf{R}^n$ and all $t \in (0,1)$, the inequality $\rho((1-t)\,x + ty) \geqq \rho(x)^{1-t}\rho(y)^t$ holds. Up to the set of the Lebesgue measure zero, we can represent such a function in the form $\rho(x) = e^{-U(x)}$, $x \in K$, where U is a convex function on some open convex set $K \subset \mathbf{R}^n$, setting $\rho = 0$ outside of K (in this representation ρ is semicontinuous from below). In particular, Theorem 1 can be applied to the normalized contraction of the measure γ_n onto the arbitrary convex subset $\mathbf{R}^n$ of the positive measure.

Note that using (1) we obtain inequality (2) for each image $\mu = \gamma_n T^{-1}$ of the measure γ_n under the map $T\colon\ \mathbf{R}^n \to \mathbf{R}^n$ having a finite Lipschitz seminorm $c = \|T\|_{\mathrm{Lip}}$. For example, as was noted in [16], the uniform distribution on a cube $[0,1]^n$ can be obtained from γ_n with the help of the map T with the Lipschitz seminorm $c = 1/\sqrt{2\pi}$. The uniform distribution on a ball of volume 1 (hence of radius $\sqrt{n}$) can also be obtained in the same way and the respective Lipschitz seminorm will be bounded by a constant independent of the dimensionality [8]. Continuing the analogy it will be interesting to see whether or not one can obtain the Bakry–Ledoux theorem in the same way (the problem of the intrinsic characterization of a class of all Lipschitz images of the Gaussian measure has not yet been solved).

There exists another way that permits us to obtain Theorem 1 and the Gaussian isoperimetric inequality as a direct corollary of the general localization lemma of Lovász and Simonovits [23]. The localization method in the form of [23] and [17] permits us to reduce some relations between multidimensional integrals to the relations between one-dimensional integrals, and, as we shall show in this paper, it particularly concerns inequality (2) (similar ideas were developed earlier by Gromov and Milman who investigated isoperimetric inequalities on a sphere; see [15], [1]). In the one-dimensional case probability measures having a log-concave density with respect to γ_1 are indeed Lipschitz images of γ_1, and thus we can apply the one-dimensional inequality (1).

Denote by $\mathcal{F}_n$ a family of all functions $f\colon\ \mathbf{R}^n \to \mathbf{R}$ such that $\|f\|_{\mathrm{Lip}} \leqq 1$. Inequality (2) admits a series of equivalent statements in terms of distributions of Lipschitz functions in the same way as (1). For example, under the conditions of Theorem 1 we can equivalently affirm that for any $f \in \mathcal{F}_n$ there exists a nondecreasing function $f^* \in \mathcal{F}_1$ having the same distribution with respect to γ_1 as f has with respect to μ: $\mu f^{-1} = \gamma_1(f^*)^{-1}$. From here, in particular, follows an inequality for probabilities of deviations of f from its median $m(f)$:

$$\mu\bigl\{f - m(f) \geqq h\bigr\} \leqq 1 - \Phi(h), \qquad h > 0.$$

We can write, with little degradation of the right-hand side, an analogous estimate for probability deviations of f from its mean value $\mathbf{E}_\mu f = \int f\,d\mu$. Cirel'son, Ibragimov, and Sudakov [16] were the first to study this problem in the Gaussian case $\mu = \gamma_n$; they proved the inequality

$$\gamma_n\{f - \mathbf{E}_{\gamma_n} f \geqq h\} \leqq 2(1 - \Phi(h)), \qquad h > 0. \tag{3}$$

The approach proposed in [16] is purely Gaussian. It is based on the representation of $f - \mathbf{E}_{\gamma_n} f$ in the form of a functional of a Brownian process with random time and does not use Gaussian isoperimetric inequality. For large h, inequality (3) can be improved with the help of one Gaussian correlation identity (see [7]):

$$\gamma_n\{f - \mathbf{E}_{\gamma_n} f \geqq h\} \leqq \frac{\varphi(h)}{h}, \qquad h > 0.$$

Indeed, relying on (1) and (2), one can get the exact information.

Corollary 1. *Let μ be a probability measure on $\mathbf{R}^n$ having a log-concave density with respect to the measure γ_n. Then*

$$\sup_{f\in\mathcal{F}_n} \mu\{f-\mathbf{E}_\mu f \geqq h\} \leqq \sup_{f\in\mathcal{F}_n} \gamma_n\{f-\mathbf{E}_{\gamma_n} f \geqq h\} = 1-\Phi(\alpha(h)), \qquad h>0,$$

where $\alpha=\alpha(h)$ is a unique root of $\alpha\,\Phi(\alpha)+\varphi(\alpha)=h$.

Obviously, $\alpha(h)\to-\infty$ for $h\to 0+$, so the constant 2 in (3) cannot be improved.

Let us note one more corollary of Theorem 1. Let us introduce the identity function $f_1(x)=x$ on $\mathbf{R}$.

Corollary 2. *Let μ be a probability measure on $\mathbf{R}^n$ having a log-concave density with respect to γ_n, and let Ψ be a convex function on $\mathbf{R}$. Then for all $f\in\mathcal{F}_n$ with mean $\mathbf{E}_\mu f=0$*

$$\mathbf{E}_\mu \Psi(f) \leqq \mathbf{E}_{\gamma_1} \Psi(f_1). \tag{4}$$

In the Gaussian case $\mu=\gamma_n$ such comparative estimates have also been studied by many authors. In the case $\Psi(x)=x^2$ inequality (4) follows from an inequality of the Poincaré type for a Gaussian measure. In the case $\Psi(x)=e^{\lambda x}$ we arrive at the other known inequality

$$\mathbf{E}_{\gamma_n} e^{\lambda f} \leqq e^{\lambda^2/2}, \qquad \lambda\in\mathbf{R}. \tag{5}$$

Although first obtained by Pisier and Maurey (see [25]), who applied, as in [16], functionals of a Brownian process, Ledoux [18] found another proof based on the properties of Ornstein–Uhlenbeck operators. Later it was discovered [19], [20] that (5) follows from the logarithmic Gross inequality. Under the additional assumption of a convexity of the function f, inequality (5) follows from one exponential inequality of Tsirel'son for a Gaussian random process [27], [28]. The case of an arbitrary convex function Ψ in the inequality

$$\mathbf{E}_{\gamma_n} \Psi(f) \leqq \mathbf{E}_{\gamma_1} \Psi(f_1) \tag{6}$$

was considered by Pinelis [24]. He modified the approach proposed in [25] and obtained the more general inequality for the case where the Lipschitz function f takes values in $\mathbf{R}^k$, and Ψ is defined and convex on $\mathbf{R}^k$ (respectively, the measure γ_1 must be replaced with γ_k). In [24] it is also shown that (6) implies (3) with a constant e instead of 2 and that this constant cannot be improved using (6). So inequalities for the deviations from the mean values are not equivalent to inequalities for moments in a class of convex functions Ψ. We give a simple proof of (6) using the Gaussian isoperimetric inequality.

Before we begin the proof let us formulate the localization Lovász–Simonovits theorem mentioned above.

Lemma 1. *Let continuous-from-below summable functions u,v on $\mathbf{R}^n$ be given such that*

$$\int_{\mathbf{R}^n} u(x)\,dx>0, \quad \int_{\mathbf{R}^n} v(x)\,dx>0. \tag{7}$$

Then there are $a,b\in\mathbf{R}^n$ and an affine function $\ell\colon (0,1)\to(0,+\infty)$ such that

$$\int_0^1 u((1-t)\,a+tb)\,\ell(t)^{n-1}\,dt>0, \quad \int_0^1 v((1-t)\,a+tb)\,\ell(t)^{n-1}\,dt>0. \tag{8}$$

The last two integrals can be considered as normalized n-dimensional integrals of u and v over a frustum of a cone with an infinitely small base. In (7)–(8) (or, for example, only in (8)) one can replace symbols of strong inequalities with symbols of weak inequalities.

The proof of Lemma 1 given in [23] consists of two steps. In the first step the decreasing sequence of compacts K_i contracting in a point or interval $[a,b]\subset\mathbf{R}^n$ is constructed so that the normalized integrals

$$\frac{1}{\mathrm{Vol}_n(K_i)}\int_{K_i} u(x)\,dx, \quad \frac{1}{\mathrm{Vol}_n(K_i)}\int_{K_i} v(x)\,dx$$

remain positive.

These integrals converge to one-dimensional integrals but, by the Brunn–Minkowski inequality, we pass in limit to (8) with some positive concave function ℓ on (0,1). Note that in this case the function $\psi(t) = \ell(t)^{n-1}$ continued by zero outside of $(0,1)$ is bounded and log-concave. We apply Lemma 1 in this weakened variant.

Proof of Theorem 1. First we show how Lemma 1 reduces Theorem 1 to the dimensionality $n = 1$. With respect to the Lebesgue measure, we can write the density of the measure μ in the form

$$\frac{d\mu(x)}{dx} = \rho(x)\,\varphi_n(x), \qquad x \in \mathbf{R}^n,$$

where $\varphi_n(x) = d\gamma_n(x)/dx = (2\pi)^{-n/2}\,e^{-|x|^2/2}$ and ρ is the log-concave continuous-from-below function on $\mathbf{R}^n$. Fix $p \in (0,1)$, $h > 0$, and choose an arbitrary open set A in $\mathbf{R}^n$ of the measure $\mu(A) > p$. Using the one-dimensional inequality of Theorem 1, we show that $\mu(A^h) \geqq \Phi(\Phi^{-1}(p) + h)$. Supposing the opposite we introduce the functions

$$u(x) = \big(\Phi(\Phi^{-1}(p) + h) - 1_{A^h}(x)\big)\,\rho(x)\,\varphi_n(x), \quad v(x) = \big(1_A(x) - p\big)\,\rho(x)\,\varphi_n(x),$$

where 1_A denotes the indicator function. The functions u and v are continuous from below and $\int_{\mathbf{R}^n} u(x)\,dx > 0$, $\int_{\mathbf{R}^n} v(x)\,dx > 0$. By Lemma 1 there exist $a, b \in \mathbf{R}^n$ and a bounded log-concave function $\psi\colon \mathbf{R} \to [0, +\infty)$ such that

$$\int_0^1 u\big((1-t)\,a + tb\big)\,\psi(t)\,dt > 0, \quad \int_0^1 v\big((1-t)\,a + tb\big)\,\psi(t)\,dt > 0.$$

The assumption $a = b$ leads to a contradiction. In the case $a \neq b$, we set $\theta = (b-a)/|b-a|$, $r = |b-a|$ and after a change of variable $z = rt$ we rewrite the last two inequalities in the form

$$\int_0^r u(a + z\theta)\,w(z)\,dz > 0, \quad \int_0^r v(a + z\theta)\,w(z)\,dz > 0, \tag{9}$$

where $w(z) = \psi(z/r)$.

Introduce a probability measure ν on $\mathbf{R}$ concentrated in $[0, r]$ and having on it the density

$$\frac{d\nu(z)}{dz} = \frac{1}{c}\,w(z)\,\rho(a + z\theta)\,\varphi_n(a + z\theta), \qquad z \in [0, r],$$

where $c = \int_0^1 w(y)\rho(a + y\theta)\,\varphi_n(a + y\theta)\,dy$ is a normalizing factor. Since $|\theta| = 1$, the function $\varphi_n(a + z\theta)/\varphi(z)$ is log-concave with respect to $z \in \mathbf{R}$, and thus ν has a log-concave density with respect to the measure γ_1. Now we introduce sets on the line

$$B = \{z \in \mathbf{R}\colon\ a + z\theta \in A\}, \quad C = \{z \in \mathbf{R}\colon\ a + z\theta \in A^h\}.$$

Then the inequalities in (9) become

$$\nu(C) < \Phi\big(\Phi^{-1}(p) + h\big), \qquad \nu(B) > p. \tag{10}$$

By construction, $B^h \subset C$ (here the h-neighborhood is considered on the line); hence $\nu(B^h) \leqq \nu(C)$. In view of (10) we obtain $\nu(B^h) < \Phi(\Phi^{-1}(\nu(B)) + h)$, which contradicts the one-dimensional inequality of Theorem 1. So

$$\mu(A^h) \geqq \Phi\big(\Phi^{-1}(\mu(A)) + h\big) \tag{11}$$

for all open, and hence all measurable, $A \subset \mathbf{R}^n$ as soon as we obtain that inequality for $n = 1$.

Now we consider the one-dimensional case. Let μ be a probability measure on $\mathbf{R}$ having with respect to the Lebesgue measure a density of the form

$$f(x) = \rho(x)\,\varphi(x), \tag{12}$$

where ρ is a log-concave function on $\mathbf{R}$. Let $F(x) = \mu(-\infty, x]$ be a distribution function of the measure μ. The measure μ is concentrated on some interval (α, β), possibly infinite, inside which ρ is positive and continuous. Let $F^{-1}\colon (0,1) \to (\alpha, \beta)$ be inverse to F. Show that the continuously differentiable increasing map $T(x) = F^{-1}(\Phi(x))$ transforming the measure γ_1 into μ has the Lipschitz seminorm $\|T\|_{\mathrm{Lip}} \leqq 1$. The last is equivalent to the inequality $f(F^{-1}(p)) \geqq \varphi(\Phi^{-1}(p))$ for all $p \in (0,1)$.

For this we prove the stronger statement: If a positive finite measure μ on $\mathbf{R}$ has a density of the form (12) with a log-concave function ρ and if a point $x_0 \in \mathbf{R}$ is such that $\mu(-\infty, x_0) \geqq p$, $\mu(x_0, +\infty) \geqq q$ (where $p \in (0,1)$ is fixed and $q = 1-p$), then

$$f(x_0) \geqq \varphi\bigl(\Phi^{-1}(p)\bigr). \tag{13}$$

Represent ρ in the form $\rho(x) = e^{-U(x)}$, where U is a convex function finite in some interval (α, β) of the full μ-measure. Then $x_0 \in (\alpha, \beta)$ and we can construct a tangent ℓ to U at point x_0. The function $\rho_0(x) = e^{-\ell(x)}$ is log-concave and on the real axis satisfies inequality $\rho_0 \geqq \rho$, and a measure μ_0 on $\mathbf{R}$ defined with respect to the Lebesgue measure by the density $f_0(x) = \rho_0(x)\,\varphi(x)$ satisfies $\mu_0(-\infty, x_0) \geqq p$, $\mu_0(x_0, +\infty) \geqq q$. Moreover, $f_0(x_0) = f(x_0)$. Hence, without loss of generality we can assume in what follows (proving our stronger assumption) that $U = \ell$ is an affine function on $(\alpha, \beta) = \mathbf{R}$.

So let $f(x) = Ce^{\lambda x}\varphi(x)$ with parameters $C > 0$, $\lambda \in \mathbf{R}$. We have

$$\mu(-\infty, x_0) = \int_{-\infty}^{x_0} f(x)\,dx = Ce^{\lambda^2/2}\,\Phi(x_0 - \lambda) \geqq p,$$
$$\mu(x_0, +\infty) = \int_{x_0}^{+\infty} f(x)\,dx = Ce^{\lambda^2/2}\bigl(1 - \Phi(x_0 - \lambda)\bigr) \geqq q$$

if and only if $C \geqq e^{-\lambda^2/2}\max\{p/\Phi(x_0-\lambda), q/(1-\Phi(x_0-\lambda))\}$. Hence

$$\begin{aligned} f(x_0) &\geqq e^{-\lambda^2/2}\max\left\{\frac{p}{\Phi(x_0-\lambda)},\ \frac{q}{1-\Phi(x_0-\lambda)}\right\} e^{\lambda x_0}\varphi(x_0) \\ &= \max\left\{\frac{p}{\Phi(x_0-\lambda)},\ \frac{q}{1-\Phi(x_0-\lambda)}\right\}\varphi(x_0-\lambda). \end{aligned}$$

Since λ may be arbitrary in view of (13) we have to show that for all $z \in \mathbf{R}$

$$\max\left\{\frac{p}{\Phi(z)}, \frac{q}{1-\Phi(z)}\right\}\varphi(z) \geqq \varphi\bigl(\Phi^{-1}(p)\bigr). \tag{14}$$

For $z \leqq \Phi^{-1}(p)$ the left-hand side of (14) is of the form $(p/\Phi(z))\,\varphi(z)$. Being a function of z it is decreasing (since $\log\Phi$ is concave) and thus it takes the minimal value at the extreme point $z = \Phi^{-1}(p)$. Analogously, at the same point the left-hand side of (14) is minimizing on the interval $z \geqq \Phi^{-1}(p)$. It remains to remark that for $z = \Phi^{-1}(p)$ inequality (14) becomes the equality.

So the map T is a contraction of the measure γ_1, and thus one-dimensional isoperimetric inequality (11) holds for all μ under consideration as soon as it holds for the measure $\mu = \gamma_1$. This concrete case can also be easily verified (see, for example, [4]). Theorem 1 is proved.

Proof of Corollary 1. As was noted, Theorem 1 can be formulated in the following way: For any function $f \in \mathcal{F}_n$ there exists a nondecreasing function $f^* \in \mathcal{F}_1$ such that $\mu f^{-1} = \gamma_1(f^*)^{-1}$. Thus

$$\sup_{f\in\mathcal{F}_n}\mu\{f - \mathbf{E}_\mu f \geqq h\} \leqq \sup_{f\in\mathcal{F}_n}\gamma_n\{f - \mathbf{E}_{\gamma_n} f \geqq h\} = \sup_{f\in\mathcal{F}_1^+}\gamma_1\{f - \mathbf{E}_{\gamma_1} f \geqq h\},$$

where $\mathcal{F}_1^+$ denotes a family of all nondecreasing functions f on $\mathbf{R}$ with the Lipschitz seminorm $\|f\|_{\mathrm{Lip}} \leqq 1$.

We maximize $\gamma_1\{f - \mathbf{E}_{\gamma_1} f \geqq h\}$ in the class $\mathcal{F}_1^+$. Let $f \in \mathcal{F}_1^+$, $a = \mathbf{E}_{\gamma_1} f$. If $h > f(x) - a$ for all $x \in \mathbf{R}$, then there is nothing to prove. Otherwise there exists a minimal $\alpha \in \mathbf{R}$ such that $f(\alpha) = a + h$. Remark that the function $g(x) = \min\{x - \alpha, 0\} + a + h$ belongs to $\mathcal{F}_1^+$

and satisfies the inequality $g(x) \leqq f(x)$ for all $x \in \mathbf{R}$. In particular, $\mathbf{E}_{\gamma_1} g \leqq a$. Moreover, since $f(x) < f(\alpha)$ for $x < \alpha$, it follows that $\{f \geqq a + h\} = \{g \geqq a + h\} = [\alpha, +\infty)$. Hence,

$$\gamma_1\{f - a \geqq h\} = \gamma_1\{g - a \geqq h\} \leqq \gamma_1\{g - \mathbf{E}_{\gamma_1} g \geqq h\}.$$

So in view of the fact that the probabilities under consideration do not change after adding constants to the functions, we can restrict ourselves to a class of functions of the form $g(x) = \min\{x - \alpha, 0\}$. For such functions we have

$$\mathbf{E}_{\gamma_1} g = \int_{-\infty}^{\alpha} (x - \alpha)\, d\Phi(x) = -\varphi(\alpha) - \alpha\Phi(\alpha) \equiv -\xi(\alpha).$$

Thus

$$\gamma_1\{g - \mathbf{E}_{\gamma_1} g \geqq h\} = 1 - \Phi\big(h + \alpha - \xi(\alpha)\big) \qquad \text{if} \quad h \leqq \xi(\alpha), \tag{15}$$

and $\gamma_1\{g - \mathbf{E}_{\gamma_1} g \geqq h\} = 0$ in the case $h > \xi(\alpha)$. Further we have $\xi'(\alpha) = \Phi(\alpha)$, and hence the function $\alpha - \xi(\alpha)$ is continuously increasing on the real axis changing in the interval $(-\infty, \ +\infty)$. Thus the right-hand side in (15) is maximal in the case $\xi(\alpha) = h$. This proves Corollary 1.

Proof of Corollary 2. As above, Theorem 1 reduces Corollary 2 to the case where $n = 1$, $\mu = \gamma_1$, and $f \in \mathcal{F}_1^+$. As the following statement shows, here the Gaussian property is unessential.

LEMMA 2. *Let μ be a probability measure on $\mathbf{R}$ with a finite first moment and let Ψ be a convex function on $\mathbf{R}$. Then in the class of all nondecreasing functions f on $\mathbf{R}$ with the Lipschitz seminorm $\|f\|_{\mathrm{Lip}} \leqq 1$ the expression $\mathbf{E}_\mu \Psi(f - \mathbf{E}_\mu f)$ achieves its maximal value* (*possibly, infinite*) *on the identity function $f = f_1$.*

Proof. Set $u = f - \mathbf{E}_\mu f$, $u_1 = f_1 - \mathbf{E}_\mu f_1$. We can assume that the function Ψ is differentiable everywhere and its derivative Ψ' is bounded. In this case the function $\psi(t) = \mathbf{E}_\mu \Psi((1-t)\, u + t u_1)$ is finite everywhere, differentiable, and

$$\psi'(t) = \mathbf{E}_\mu \Psi'\big((1-t)\, u + t u_1\big)(u_1 - u), \qquad t \in \mathbf{R}.$$

Supposing the opposite we assume that $\mathbf{E}_\mu \Psi(u_1) < \mathbf{E}_\mu \Psi(u)$, i.e., that $\psi(1) < \psi(0)$. Since ψ is convex, this assumption implies the inequality $\psi'(0) < 0$, i.e.,

$$\mathbf{E}_\mu \Psi'(u)\, u_1 < \mathbf{E}_\mu \Psi'(u)\, u. \tag{16}$$

On the other hand, taking into account that $\mathbf{E}_\mu u = \mathbf{E}_\mu u_1 = 0$, u satisfies $0 \leqq u(x) - u(y) \leqq x - y = u_1(x) - u_1(y)$ for all $x > y$, and Ψ' is not decreasing, we obtain

$$\begin{aligned}\mathbf{E}_\mu \Psi'(u)\, u &= \iint_{x>y} \big(\Psi'(u(x)) - \Psi'(u(y))\big)\, (u(x) - u(y))\, d\mu(x)\, d\mu(y) \\ &\leqq \iint_{x>y} \big(\Psi'(u(x)) - \Psi'(u(y))\big)\, \big(u_1(x) - u_1(y)\big)\, d\mu(x)\, d\mu(y) = \mathbf{E}_\mu \Psi'(u)\, u_1.\end{aligned}$$

However, this contradicts (16). Lemma 2 and Corollary 2 are proved.

Remark. After this paper had been prepared, the author learned about the paper [29]. It specifically proves that all the probability measures considered in Theorem 1 can be represented as Lipschitz images of the standard Gaussian measure in $\mathbf{R}^n$.

REFERENCES

[1] S. ALESKER, *Localization technique on the sphere and the Gromov–Milman theorem on the concentration phenomenon on uniformly convex sphere*, Math. Sci. Res. Inst. Publ., 34 (1996), pp. 17–27.

[2] D. BAKRY AND M. LEDOUX, *Lévy–Gromov's isoperimetric inequality for an infinite dimensional diffusion generator*, Invent. Math., 123 (1996), pp. 259–281.

[3] F. Barthe and B. Maurey, *Some remarks on isoperimetry of Gaussian type*, Ann. Inst. H. Poincaré Probab. Statist., 36 (2000), pp. 419–434.

[4] S. G. Bobkov, *An isoperimetric problem on the line*, J. Math. Sci. (New York), 88 (1998), pp. 3–6.

[5] S. G. Bobkov, *An isoperimetric inequality on the discrete cube, and an elementary proof of the isoperimetric inequality in Gauss space*, Ann. Probab., 25 (1997), pp. 206–214.

[6] S. G. Bobkov and F. Götze, *Discrete isoperimetric and Poincaré-type inequalities*, Probab. Theory Related Fields, 114 (1999), pp. 245–277.

[7] S. G. Bobkov, F. Götze, and C. Houdré, *On Gaussian and Bernoulli covariance representations*, Bernoulli, 7 (2001), pp. 439–451.

[8] S. G. Bobkov and M. Ledoux, *From Brunn–Minkowski to Brascamp–Lieb and to logarithmic Sobolev inequalities*, Geom. Funct. Anal., 10 (2000), pp. 1028–1052.

[9] V. I. Bogachev, *Gaussian Measures*, Math. Surveys and Monographs 62, AMS, Providence, RI, 1998.

[10] C. Borell, *The Brunn–Minkowski inequality in Gauss space*, Invent. Math., 30 (1975), pp. 207–216.

[11] Yu. A. Davydov, M. A. Lifshits, and N. V. Smorodina *Local Properties of Distributions of Stochastic Functionals*, Transl. Math. Monographs 173, AMS, Providence, RI, 1998.

[12] A. Ehrhard, *Symétrisation dans l'espace de Gauss*, Math. Scand., 53 (1983), pp. 281–301.

[13] M. Gromov, *Paul Lévy's Isoperimetric Inequality*, Preprint, Institut des Hautes Études Scientifiques, Paris, 1980.

[14] M. Gromov, *Large Riemannian manifolds*, in Curvature and Topology of Riemannian Manifolds (Katata, 1985), Lecture Notes in Math. 1201, K. Shiohama, T. Sakai, and T. Sunada, eds., Springer-Verlag, Berlin, 1986, pp. 108–121.

[15] M. Gromov and V. D. Milman, *Generalization of the spherical isoperimetric inequality to uniformly convex Banach spaces*, Compositio Math., 62 (1987), pp. 263–282.

[16] B. S. Cirel'son, I. A. Ibragimov, and V. N. Sudakov, *Norms of Gaussian sample functions*, in Proceedings of the Third Japan–USSR Symposium on Probability Theory, Lecture Notes in Math. 550, Springer-Verlag, Berlin, 1976, pp. 20–41.

[17] R. Kannan, L. Lovász, and M. Simonovits, *Isoperimetric problems for convex bodies and a localization lemma*, Discrete Comput. Geom., 13 (1995), pp. 541–559.

[18] M. Ledoux, *Semigroup proofs of the isoperimetric inequality in Euclidean and Gauss space*, Bull. Sci. Math., 118 (1994), pp. 485–510.

[19] M. Ledoux, *Isoperimetry and Gaussian Analysis*, in Lectures on Probability Theory and Statistics (Saint-Flour, 1994), Lecture Notes in Math. 1648, Springer-Verlag, Berlin, 1996, pp. 165–294.

[20] M. Ledoux, *Remarks on logarithmic Sobolev constants, exponential integrability and bounds on the diameter*, J. Math. Kyoto Univ., 35 (1995), pp. 211–220.

[21] M. Ledoux, *Concentration of measure and logarithmic Sobolev inequalities*, in Séminaire de Probabilités, XXXIII, Lecture Notes in Math. 1709, Springer-Verlag, Berlin, 1999, pp. 120–216.

[22] M. A. Lifshits, *Gaussian Stochastic Functions*, TViMS, Kiev, 1995 (in Russian).

[23] L. Lovász and M. Simonovits, *Random walks in a convex body and an improved volume algorithm*, Random Structures Algorithms, 4 (1993), pp. 359–412.

[24] I. F. Pinelis, *Fractional sums and integrals of r-concave tails and applications to comparison probability inequalities*, Contemp. Math., 234 (1999), pp. 149–168.

[25] G. Pisier, *Probabilistic methods in the geometry of Banach spaces*, in Probability and Analysis, Lecture Notes in Math. 1206, Springer-Verlag, Berlin, 1986, pp. 167–241.

[26] V. N. Sudakov and B. S. Tsirel'son *Extremal properties of half-spaces for spherically invariant measures*, in Problems in the Theory of Probability Distributions. II, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 41 (1974), pp. 14–24 (in Russian).

[27] B. S. Tsirel'son, *A geometric approach to maximum likelihood estimation for infinite-dimensional Gaussian location.* II, Theory Probab. Appl., 30 (1985), pp. 820–828.

[28] B. S. Tsirel'son, *A geometric approach to maximum likelihood estimation for infinite-dimensional location.* III, Theory Probab. Appl., 31 (1986), pp. 470–483.

[29] L. A. Caffarelli, *Monotonicity properties of optimal transportation and the FKG and related inequalities*, Comm. Math. Phys., 214 (2000), pp. 547–563.