On the Central Limit Property of Convex Bodies

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Summary. For isotropic convex bodies K in $\mathbf{R}^n$ with isotropic constant L_K, we study the rate of convergence, as n goes to infinity, of the average volume of sections of K to the Gaussian density on the line with variance L_K^2.

Let K be an isotropic convex body in $\mathbf{R}^n$, $n \geq 2$, with volume one. By the isotropy assumption we mean that the baricenter of K is at the origin, and there exists a positive constant L_K so that, for every unit vector θ,

$$\int_K \langle x, \theta \rangle^2 \, dx = L_K^2.$$

Introduce the function

$$f_K(t) = \int_{S^{n-1}} \mathrm{vol}_{n-1}\big(K \cap H_\theta(t)\big) \, d\sigma(\theta), \quad t \in \mathbf{R},$$

expressing the average $(n-1)$-dimensional volume of sections of K by hyperplanes $H_\theta(t) = \{x \in \mathbf{R}^n : \langle x, \theta \rangle = t\}$ perpendicular to $\theta \in S^{n-1}$ at distance $|t|$ from the origin (and where σ is the normalized uniform measure on the unit sphere).

When the dimension n is large, the function f_K is known to be very close to the Gaussian density on the line with mean zero and variance L_K^2. Being general and informal, this hypothesis needs to be formalized and verified, and precise statements may depend on certain additional properties of convex bodies. For some special bodies K, several types of closeness of f_K to Gaussian densities were recently studied in [B-V], cf. also [K-L]. To treat the general case, the following characteristic σ_K^2 associated with K turns out to be crucial:

$$\sigma_K^2 = \frac{\mathrm{Var}(|X|^2)}{n L_K^4}.$$

Here X is a random vector uniformly distributed over K, and $\mathrm{Var}(|X|^2)$ denotes the variance of $|X|^2$. In particular, we have the following statement which is proved in this note.

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Theorem 1. *For all* $0 < |t| \le c\sqrt{n}$,

$$\left| f_K(t) - \frac{1}{\sqrt{2\pi} L_K} e^{-t^2/(2L_K^2)} \right| \le C \left[\frac{\sigma_K L_K}{t^2 \sqrt{n}} + \frac{1}{n} \right] , \tag{1}$$

where c *and* C *are positive numerical constants.*

Using Bourgain's estimate $L_K \le c \log(n)\, n^{1/4}$ ([Bou], cf. also [D], [P]) the right-hand side of (1) can be bounded, up to a numerical constant, by

$$\frac{\sigma_K \log n}{t^2 n^{1/4}} + \frac{1}{n},$$

which is small for large n up to the factor σ_K. Let us look at the behavior of this quantity in some canonical cases.

For the n-cube $K = [-\frac{1}{2}, \frac{1}{2}]^n$, by the independence of coordinates, $\sigma_K^2 = \frac{4}{5}$.

For K's the normalized ℓ_1^n balls,

$$\sigma_K^2 = 1 - \frac{2(n+1)}{(n+3)(n+4)} \to 1, \quad \text{as} \quad n \to \infty.$$

Normalization condition refers to $\mathrm{vol}_n(K) = 1$, but a slightly more general definition $\sigma_K^2 = \frac{n \mathrm{Var}(|X|^2)}{(\mathbf{E}|X|^2)^2}$ makes this quantity invariant under homotheties and simplifies computations.

For K's the normalized Euclidean balls,

$$\sigma_K^2 = \frac{4}{n+4} \to 0, \quad \text{as} \quad n \to \infty.$$

Thus, σ_K^2 can be small and moreover, in the space of any fixed dimension, the Euclidean balls provide the minimum (cf. Theorem 2 below).

The property that σ_K^2 is bounded by an absolute constant for all ℓ_p^n balls simultaneously was recently observed by K. Ball and I. Perissinaki [B-P] who showed for these bodies that the covariances $\mathrm{cov}(X_i^2, X_j^2) = \mathbf{E}X_i^2 X_j^2 - \mathbf{E}X_i^2 \mathbf{E}X_j^2$ are non-positive. Since in general $\mathrm{Var}(|X|^2) = \sum_{i=1}^n \mathrm{Var}(X_i^2) + \sum_{i \ne j} \mathrm{cov}(X_i^2, X_j^2)$, the above property together with the Khinchine-type inequality implies

$$\mathrm{Var}(|X|^2) \le \sum_{i=1}^n \mathrm{Var}(X_i^2) \le \sum_{i=1}^n \mathbf{E}X_i^4 \le C n L_K^4.$$

The result was used in [A-B-P] to study the closeness of random distribution functions $F_\theta(t) = \mathbf{P}\{\langle X, \theta \rangle \le t\}$, for most of θ on the sphere, to the normal distribution function with variance L_K^2. This randomized version of the central limit theorem originates in the paper by V. N. Sudakov [S], cf. also [D-F], [W]. The reader may find recent related results in [K-L], [Bob], [N-R], [B-H-V-V]. It has become clear since the work [S] that, in order to get closeness to normality, the convexity assumption does not play a crucial role, and one rather needs a dimension-free concentration of $|X|$ around its mean. Clearly, the strength of concentration can be measured in terms of the variance of $|X|^2$, for example.

Nevertheless, the question on whether or not the quantity σ_K^2 can be bounded by a universal constant in the general convex isotropic case is still open, although it represents a rather weak form of Kannan-Lovász-Simonovits' conjecture about Cheeger-type isoperimetric constants for convex bodies [K-L-S]. For isotropic K, the latter may equivalently be expressed as the property that, for any smooth function g on $\mathbf{R}^n$, for some absolute constant C,

$$\int_K \left| g(x) - \int_K g(x)\,dx \right| dx \leq C L_K \int_K |\nabla g(x)|\,dx. \tag{2}$$

By Cheeger's theorem, the above implies a Poincaré-type inequality

$$\int_K \left| g(x) - \int_K g(x)\,dx \right|^2 dx \leq 4 (C L_K)^2 \int_K |\nabla g(x)|^2\,dx$$

which for $g(x) = |x|^2$ becomes $\mathrm{Var}(|X|^2) \leq 16 n C^2 L_K^4$, that is, $\sigma_K^2 \leq 16 C^2$.

To bound an optimal C in (2), R. Kannan, L. Lovász, and M. Simonovits considered in particular the geometric characteristic

$$\chi(K) = \int_K \chi_K(x)\,dx$$

where $\chi_K(x)$ denotes the length of the longest interval lying in K with center at x. By applying the localization lemma of [L-S], they proved that (2) holds true with $C L_K = 2\chi(K)$. Therefore, $\sigma_K L_K \leq 8\chi(K)$, and thus the right-hand side of (1) can also be bounded, up to a constant, by

$$\frac{\chi(K)}{t^2\sqrt{n}} + \frac{1}{n}.$$

To prove Theorem 1, we need the following formula which also appears in [B-V, Lemma 1.2].

Lemma 1. *For all t,*

$$f_K(t) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \int_{K \cap \{|x| \geq |t|\}} \frac{1}{|x|} \left(1 - \frac{t^2}{|x|^2}\right)^{\frac{n-3}{2}} dx.$$

For completeness, we prove it below (with a somewhat different argument).

Proof. We may assume $t \geq 0$. Denote by $\lambda_{\theta,t}$ the Lebesgue measure on $H_\theta(t)$. Then

$$\lambda_t = \int_{S^{n-1}} \lambda_{\theta,t}\, d\sigma(\theta)$$

is a positive measure on $\mathbf{R}^n$ such that $f_K(t) = \lambda_t(K)$. This measure has density that is invariant with respect to rotations, i.e.,

$$\frac{d\lambda_t}{dx} = p_t(|x|),$$

where p_t is a function on $[t, \infty)$. To find the function p_t, note first that, for every $r > t$,

$$\lambda_t\big(B(0,r)\big) = \int_{B(0,r)} p_t(|x|)\, dx = |S^{n-1}| \int_t^r p_t(s) s^{n-1}\, ds,$$

where $B(0,r)$ is the Euclidean ball with center at the origin and radius r, and $|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of the sphere S^{n-1}. On the other hand, since the section of $B(0,r)$ by the hyperplane $H_\theta(t)$ is the Euclidean ball in $\mathbf{R}^{n-1}$ of radius $(r^2 - t^2)^{1/2}$, we have

$$\lambda_t\big(B(0,r)\big) = \int_{S^{n-1}} \lambda_{\theta,t}\big(B(0,r)\big)\, d\sigma(\theta) = \frac{\pi^{(n-1)/2}}{\Gamma\big(1 + (n-1)/2\big)} (r^2 - t^2)^{(n-1)/2}.$$

Taking the derivatives by r, we see that for every $r \geq t$,

$$\frac{n-1}{2} (r^2 - t^2)^{(n-1)/2}\, 2r = \frac{2\pi^{1/2}\, \Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\, p_t(r) r^{n-1},$$

which implies

$$p_t(r) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\, \Gamma\left(\frac{n-1}{2}\right)} \frac{(r^2 - t^2)^{(n-3)/2}}{r^{n-2}}.$$

Since $f_K(t) = \lambda_t(K)$, the result follows.

Proof of Theorem 1. Let $t > 0$. By the Cauchy-Schwarz inequality,

$$\int_K \big| |x|^2 - nL_K^2 \big|\, dx \leq \left(\int_K \big| |x|^2 - nL_K^2 \big|^2\, dx \right)^{1/2} = \sqrt{n}\, \sigma_K L_K^2,$$

so

$$\int_K \big| |x| - \sqrt{n} L_K \big|\, dx \;=\; \int_K \frac{\big| |x|^2 - nL_K^2 \big|}{|x| + \sqrt{n} L_K}\, dx \;\leq\; \sigma_K L_K. \tag{3}$$

By Stirling's formula,

$$\lim_{n \to \infty} \frac{\sqrt{2\pi}}{\sqrt{n}} \frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma\big((n-1)/2\big)} = 1$$

so that the constants $c_n = \frac{\Gamma(n/2)}{\sqrt{\pi}\,\Gamma((n-1)/2))}$ appearing in Lemma 1 are $O(\sqrt{n})$.

Now, on the interval $[t,\infty)$ consider the function

$$g_n(z) = \frac{1}{z}\left(1 - \frac{t^2}{z^2}\right)^{(n-3)/2}.$$

Its derivative

$$g_n'(z) \;=\; \frac{t^2(n-3)}{z^4}\left(1 - \frac{t^2}{z^2}\right)^{(n-5)/2} - \frac{1}{z^2}\left(1 - \frac{t^2}{z^2}\right)^{(n-3)/2}$$

represents the difference of two non-negative terms. Both of them are equal to zero at t, tend to zero at infinity and each has one critical point, the first at $z = t\sqrt{n-1}/2$, and the second at $z = t\sqrt{n-1}/\sqrt{2}$. Therefore,

$$\max_{z\in[t,\infty)} |g_n'(z)| \le \frac{16}{t^2(n-1)}.$$

This implies that, for every $x \in K$, $|x| \ge t$, if $\sqrt{n}L_K \ge t$, then

$$|g_n(|x|) - g_n(\sqrt{n}L_K)| \le \frac{16}{t^2(n-1)}\left||x| - \sqrt{n}L_K\right|,$$

and by (3),

$$\int_{K_t} |g_n(|x|) - g_n(\sqrt{n}L_K)|\,dx \le \frac{16\sigma_K L_K}{t^2(n-1)}, \tag{4}$$

where $K_t = K \cap \{|x| \ge t\}$.

Now, writing

$$\begin{aligned} f_K(t) &= c_n \int_{K_t} g_n(|x|)\,dx \\ &= c_n g_n(\sqrt{n}L_K)\mathrm{vol}_n(K_t) + c_n \int_{K_t} \big(g_n(|x|) - g_n(\sqrt{n}L_K)\big)\,dx \end{aligned}$$

and applying (4), we see that, for all $t \le \sqrt{n}L_K$,

$$|f_K(t) - c_n g_n(\sqrt{n}L_K)\mathrm{vol}_n(K_t)| \le \frac{C\sigma_K L_K}{t^2\sqrt{n}},$$

where C is a numerical constant. This gives

$$\big|f_K(t) - c_n g_n(\sqrt{n}L_K)\big| \le c_n g_n(\sqrt{n}L_K)\big(1 - \mathrm{vol}_n(K_t)\big) + \frac{C\sigma_K L_K}{t^2\sqrt{n}}. \tag{5}$$

Recall that $L_K \ge c$, for some universal $c > 0$ (the worst situation is attained at Euclidean balls, cf. eg. [Ba]). Therefore (5) is fulfilled under $t \le c\sqrt{n}$.

To further bound the first term on the right-hand side of (5), note that $g_n(z) \le 1/z$, so $c_n g_n(\sqrt{n}L_K) \le C_0$, for some numerical C_0. Also, if $t \le c\sqrt{n}$,

$$1 - \mathrm{vol}_n(K_t) \le \mathrm{vol}_n\big(B(0,t)\big) = \omega_n t^n \le \left(\frac{c_0}{\sqrt{n}}\right)^n \big(c\sqrt{n}\big)^n < 2^{-n},$$

where ω_n denotes the volume of the unit ball in $\mathbf{R}^n$, and where $c_0 c$ can be made less than 1/2 by choosing a proper c. This also shows that the first term in (5) will be dominated by the second one. Indeed, the inequality $C_0 2^{-n} \le \frac{C\sigma_K L_K}{t^2\sqrt{n}}$ immediately follows from $t \le c\sqrt{n}$ and the lower bound on σ_K given in Theorem 2.

Thus,

$$\big|f_K(t) - c_n g_n(\sqrt{n}L_K)\big| \le \frac{C\sigma_K L_K}{t^2\sqrt{n}},$$

and we are left with the task of comparing $c_n g_n(\sqrt{n}L_K)$ with the Gaussian density on the line. This is done in the following elementary

Lemma 2. *If $0 \le t \le \sqrt{n}L_K$, for some absolute C,*

$$\left|\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}\left(1 - \frac{t^2}{nL_K^2}\right)^{(n-3)/2}\frac{1}{\sqrt{n}L_K} - \frac{1}{\sqrt{2\pi}L_K}\,e^{-t^2/2L_K^2}\right| \le \frac{C}{n}.$$

Proof. Using the fact that L_K is bounded from below, multiplying the above inequality by $\sqrt{2\pi}L_K$ and replacing $u = t^2/(2L_K^2)$, we are reduced to estimating

$$\left|\frac{\sqrt{2}\,\Gamma\left(\frac{n}{2}\right)}{\sqrt{n}\,\Gamma\left(\frac{n-1}{2}\right)}\left(1-\frac{2u}{n}\right)^{\frac{n-3}{2}} - e^{-u}\right| \le \left|e^{-u} - \frac{\sqrt{2}\,\Gamma\left(\frac{n}{2}\right)}{\sqrt{n}\,\Gamma\left(\frac{n-1}{2}\right)}e^{-u}\right| + \frac{\sqrt{2}\,\Gamma\left(\frac{n}{2}\right)}{\sqrt{n}\,\Gamma\left(\frac{n-1}{2}\right)}\left|e^{-u} - \left(1-\frac{2u}{n}\right)^{\frac{n-3}{2}}\right|.$$

In order to estimate the first summand, use the asymptotic formula for the Γ-function, $\Gamma(x) = x^{x-1}e^{-x}\sqrt{2\pi x}\left(1 + \frac{1}{12x} + O(\frac{1}{x^2})\right)$, as $x \to +\infty$, to get

$$\frac{\sqrt{\frac{2}{n}}\,\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = \frac{\left(\frac{n}{2}\right)^{(n-3)/2} e^{-n/2}\,\sqrt{\pi n}\,\left(1 + \frac{1}{6n} + O(\frac{1}{n^2})\right)}{\left(\frac{n-1}{2}\right)^{(n-3)/2} e^{-(n-1)/2}\,\sqrt{\pi(n-1)}\left(1 + \frac{1}{6(n-1)} + O(\frac{1}{n^2})\right)}$$

$$= e^{-1/2}\left(\frac{n}{n-1}\right)^{\frac{n}{2}-1}\left(1 + O\left(\frac{1}{n^2}\right)\right).$$

Since, by Taylor, $(\frac{n}{n-1})^{\frac{n}{2}-1} = e^{(-\frac{n}{2}+1)\,\log(1-\frac{1}{n})} = e^{1/2}\left(1 + O\left(\frac{1}{n}\right)\right)$, the first summand is $O(\frac{1}{n})$ uniformly over $u \ge 0$.

To estimate the second summand, recall that $0 \le u \le n/2$. The function $\psi_n(u) = e^{-u} - \left(1 - \frac{2u}{n}\right)^{\frac{n-3}{2}}$ satisfies $\psi_n(0) = 0$, $\psi_n(n/2) = e^{-n/2}$, and the point $u_0 \in [0, n/2]$ where $\psi_n'(u_0) = 0$ (if it exists) satisfies $\left(1 - \frac{2u_0}{n}\right)^{\frac{n-5}{2}} = \frac{n}{n-3}\, e^{-u_0}$ (when $n \geq 4$). Hence, $\psi_n(u_0) = \frac{2u_0 - 3}{n-3}\, e^{-u_0} = O(\frac{1}{n})$, and thus $\sup_u \psi_n(u) = O(\frac{1}{n})$. This proves Lemma 2.

Remark. Returning to the inequality (1) of Theorem 1, it might be worthwhile to note that, in the range $|t| \geq c\sqrt{n}$, the function f_K satisfies, for some absolute $C > 0$, the estimate

$$f_K(t) \leq \frac{C}{|t|}\, e^{-t^2/(CnL_K^2)} \leq \frac{C}{c\sqrt{n}},$$

and in this sense it does not need to be compared with the Gaussian distribution in this range. Indeed, it follows immediately from the equality in Lemma 1 that

$$f_K(t) \leq C\sqrt{n}\, \max_{z \geq |t|} g_n(z)\, \mathbf{P}\{|X| \geq |t|\},$$

where X denotes a random vector uniformly distributed over K. When $n \geq 3$, in the interval $z \geq |t|$, the function $g_n(z) = \frac{1}{z}\,(1 - \frac{t^2}{z^2})^{(n-3)/2}$ attains its maximum at the point $z_0 = |t|\sqrt{n-2}$ where it takes the value $g_n(z_0) \leq \frac{1}{|t|\sqrt{n-2}}$. Hence,

$$C\sqrt{n}\, \max_{z \geq |t|} g_n(z) \leq \frac{C'}{|t|} \leq \frac{C'}{c\sqrt{n}}.$$

On the other hand, the probability $\mathbf{P}\{|X| \geq |t|\}$ can be estimated with the help of Alesker's ψ_2-estimate, [A],

$$\mathbf{E} e^{|X|^2/(C''nL_K^2)} \leq 2.$$

We finish this note with a simple remark on the extremal property of the Euclidean balls in the minimization problem for σ_K^2.

Theorem 2. $\sigma_K^2 \geq \frac{4}{n+4}$.

Proof. The distribution function $F(r) = \mathrm{vol_n}(\{x \in K : |x| \leq r\})$ of the random vector X uniformly distributed in K has density

$$F'(r) = r^{n-1} \left| S^{n-1} \cap \frac{1}{r} K \right| = |S^{n-1}|\, r^{n-1} \sigma\left(\frac{1}{r} K\right), \quad r > 0.$$

We only use the property that $q(r) = |S^{n-1}|\, \sigma(\frac{1}{r}K)$ is non-increasing in $r > 0$. Clearly, this function can also be assumed to be absolutely continuous so that we can write

$$q(r) = n \int_r^{+\infty} \frac{p(s)}{s^n}\, ds, \quad r > 0,$$

for some non-negative measurable function p on $(0, +\infty)$.

We have

$$1 = \int_0^\infty dF(r) = \int_0^\infty r^{n-1} q(r)\, dr = n \iint_{0<r<s} r^{n-1} \frac{p(s)}{s^n}\, drds = \int_0^\infty p(s)\, ds.$$

Hence, p represents a probability density of a positive random variable, say, ξ. Similarly, for every $\alpha > -n$,

$$\mathbf{E}|X|^\alpha = \int_0^\infty r^{\alpha+n-1} q(r)\, dr = \frac{n}{n+\alpha} \int_0^\infty s^\alpha p(s)\, ds = \frac{n}{n+\alpha}\, \mathbf{E}\xi^\alpha.$$

Therefore,

$$\begin{aligned} \mathrm{Var}(|X|^2) &= \frac{n}{n+4}\, \mathbf{E}\xi^4 - \left(\frac{n}{n+2}\, \mathbf{E}\xi^2\right)^2 \\ &= \frac{4n}{(n+4)(n+2)^2}\, (\mathbf{E}\xi^2)^2 + \frac{n}{n+4}\, \mathrm{Var}(\xi^2) \\ &\geq \frac{4n}{(n+4)(n+2)^2}\, (\mathbf{E}\xi^2)^2. \end{aligned}$$

One can conclude that

$$\sigma_K^2 = n\, \frac{\mathrm{Var}(|X|^2)}{(\mathbf{E}|X|^2)^2} \geq n\, \frac{\frac{4n}{(n+4)(n+2)^2}\, (\mathbf{E}\xi^2)^2}{\left(\frac{n}{n+2}\, \mathbf{E}\xi^2\right)^2} = \frac{4}{n+4}.$$

Theorem 2 follows.

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