On Convex Bodies and Log-Concave Probability Measures with Unconditional Basis*

S.G. Bobkov[1] and F.L. Nazarov[2]

[1] School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
bobkov@math.umn.edu
[2] Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA *fedja@math.msu.edu*

1 Introduction

We consider here two asymptotic properties of finite dimensional convex bodies which generate a norm with an unconditional basis. For definiteness, such a basis is taken to be the canonical basis in $\mathbf{R}^n$. Thus, assume we are given a convex set $K \subset \mathbf{R}^n$ of volume $\mathrm{vol}_n(K) = 1$ which, together with every point $x = (x_1, \dots, x_n)$, contains the parallepiped with the sides $[-|x_j|, |x_j|\,]$, $1 \leq j \leq n$. In addition, K is supposed to be in isotropic position, which is equivalent to the property that the integrals

$$\int_K x_j^2 \, dx = L_K^2, \quad 1 \leq j \leq n, \tag{1.1}$$

do not depend on j.

The isotropic constant L_K is known to satisfy $c_1 \leq L_K \leq c_2$, for some universal $c_1, c_2 > 0$. Hence, for the Euclidean norm $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$ we have

$$c_1 n \leq \int_K |x|^2 \, dx \leq c_2 n$$

and similarly, the average value of $|x|$ over K is about $\sqrt{n}$.

Consider the linear functional

$$f(x) = \frac{x_1 + \ldots + x_n}{\sqrt{n}}.$$

By (1.1), its L_2-norm over K is exactly $\|f\|_2 = L_K$. As in the case of any other linear functional, L_p-norms satisfy $\|f\|_p \leq Cp\,\|f\|_2$ for every $p \geq 1$ and some absolute C. Up to a universal constant, this property can equivalently be expressed as one inequality $\|f\|_{\psi_1} \leq C\,\|f\|_2$ for the Orlicz norm corresponding to the Young function $\psi_1(t) = e^{|t|} - 1$, $t \in \mathbf{R}$. For the concrete functional f introduced above, this can be sharpened in terms of the Young function $\psi_2(t) = e^{|t|^2} - 1$.

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Theorem 1.1. *$\|f\|_{\psi_2} \le C$, for some universal C.*

The proof might require some information on the distribution of the Euclidean norm of a point x over K. Indeed, if we observe $x = (x_1, \dots, x_n)$ as a random vector uniformly distributed in K, and if $(\varepsilon_1, \dots, \varepsilon_n)$ is an arbitrary collection of signs, then $(\varepsilon_1 x_1, \dots, \varepsilon_n x_n)$ has the same uniform distribution (by the assumption that the canonical basis is unconditional). In particular,

$$f(x, \varepsilon) = \frac{\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n}{\sqrt{n}}$$

has the same distribution as $f(x)$. But with respect to the symmetric Bernoulli measure $\mathbf{P}_\varepsilon$ on the discrete cube $\{-1, 1\}^n$, there is a subgaussian inequality

$$\mathbf{P}_\varepsilon\big\{|f(x, \varepsilon)| \ge t\big\} \le 2\, e^{-nt^2/(2|x|^2)}, \quad t \ge 0.$$

Taking the expectation over K, we arrive at

$$\mathrm{vol}_n\big\{x \in K : |f(x)| \ge t\big\} \le 2 \int_K e^{-nt^2/(2|x|^2)}\, dx. \tag{1.2}$$

This is how the distribution of the norm $|x|$ can be involved in the study of the distribution of $f(x)$. The statement of Theorem 1.1 is equivalent to the assertion that the tails of f admit a subgaussian bound

$$\mathrm{vol}_n\big\{x \in K : |f(x)| \ge t\big\} \le Ce^{-ct^2}.$$

Hence, it suffices to prove such a bound for the integral in (1.2) taken over a sufficiently big part of K. The function $e^{-nt^2/(2|x|^2)}$ under the integral sign has the desired subgaussian behaviour on the part of K where $|x|/\sqrt{n} \le \mathrm{const}$. To control large deviations of $|x|/\sqrt{n}$, we prove:

Theorem 1.2. *There exist universal $t_0 > 0$ and $c > 0$ such that, for all $t \ge t_0$,*

$$\mathrm{vol}_n\left\{x \in K : \frac{|x|}{\sqrt{n}} \ge t\right\} \le e^{-c\,t\sqrt{n}}. \tag{1.3}$$

For the "normalized" ℓ_1^n-ball, this inequality was proved by G. Schechtman and J. Zinn in [S-Z1], see also [S-Z2] for related results on deviations of the Euclidean norm and other Lipschitz functions on the ℓ_p^n-balls.

Note that too large t may be ignored in (1.3), since we always have $|x| \le Cn$, for all $x \in K$ (V.D. Milman, A. Pajor, [M-P]). Therefore, for $t > C\sqrt{n}$, the left hand side is zero. For $t \le C\sqrt{n}$, the inequality implies

$$\mathrm{vol}_n\left\{x \in K : \frac{|x|}{\sqrt{n}} \ge t\right\} \le e^{-c\,t^2/C},$$

which means that the $L_{\psi_2}(K)$-norm of the Euclidean norm is bounded by its L_2-norm, up to a universal constant. Thus, Theorem 1.2 can also be viewed as a sharpening, for isotropic convex sets with an unconditional basis, of a result of S. Alesker [A]. We do not know whether the unconditionality assumption is important for the conclusion such as (1.3). On the other hand, Theorem 1.2 as well as Theorem 1.1 (under an extra condition on the support) can be extended to all isotropic log-concave probability measures which are invariant under transformations $(x_1, \dots, x_n) \rightarrow (\pm x_1, \dots, \pm x_n)$, cf. Propositions 5.1 and 6.1 below.

Using Theorem 1.2, one may estimate the integral in (1.2) as follows:

$$\int_K e^{-n\,t^2/(2|x|^2)}\,dx = \int_{|x|\le t_0\sqrt{n}} + \int_{|x|\ge t_0\sqrt{n}} \le e^{-t^2/(2t_0^2)} + e^{-c\,t_0\sqrt{n}}$$
$$\le 2\,e^{-t^2/(2t_0^2)}$$

provided that $t \le \mathrm{const}\, n^{1/4}$. Hence, we obtain the desired subgaussian bound for relatively "small" t. To treat the values $t \ge \mathrm{const}\, n^{1/4}$, one needs to involve some other arguments which are discussed in section 6.

2 Preliminaries (the case of bodies)

Here we collect some useful, although basically known, facts about the sets K with the canonical unconditional basis as in section 1. It is reasonable to associate with K its normalized part in the positive octant $\mathbf{R}_+^n = [0, +\infty)^n$,

$$K^+ = 2K \cap \mathbf{R}_+^n.$$

Thus, if $x = (x_1, \dots, x_n)$ is viewed as a random vector uniformly distributed in K, then the vector $(2|x_1|, \dots, 2|x_n|)$ is uniformly distributed in K^+.

The set K^+ has the properties:

a) $\mathrm{vol}_n(K^+) = 1$;

b) for all $x \in K^+$ and $y \in \mathbf{R}_+^n$ with $y_j \le x_j$, $1 \le j \le n$, we have $y \in K^+$;

c) $\int_{K^+} x_j^2\,dx = 4L_K^2$, for all $1 \le j \le n$.

Proposition 2.1. $L_K^2 \le \frac{1}{2}$.

Proof. With every point $x = (x_1, \dots, x_n)$, the set K^+ contains the parallepiped $\prod_{j=1}^n [0, x_j]$. So $\prod_{j=1}^n x_j \le 1$, for every $x \in K$. Since both the sets K^+ and $V = \{x \in \mathbf{R}_+^n : \prod_{j=1}^n x_j \ge 1\}$ are convex and do not intersect each other (excluding the points on the boundaries), there exists a separating hyperplane. But any hyperplane touching the boundary of V has equation $\lambda_1 x_1 + \ldots + \lambda_n x_n = n$ with some $\lambda_j > 0$ such that $\prod_{j=1}^n \lambda_j = 1$. Therefore, $K^+ \subset \big\{x \in \mathbf{R}_+^n : \frac{\lambda_1 x_1 + \ldots + \lambda_n x_n}{n} \le 1\big\}$, and so, by the geometric-arithmetic inequality,

$$1 \geq \int_{K^+} \frac{\lambda_1 x_1 + \ldots + \lambda_n x_n}{n}\, dx \geq \bigg(\prod_{j=1}^n \int_{K^+} x_j\, dx \bigg)^{1/n}.$$

By a Khinchine-type inequality,

$$\int_{K^+} x_j\, dx \geq \frac{1}{\sqrt{2}} \bigg(\int_{K^+} x_j^2\, dx \bigg)^{1/2} = \sqrt{2}\, L_K, \tag{2.1}$$

according to the property c). Thus, $1 \geq \sqrt{2}\, L_K$.

Remark 2.1. It is a well-known fact that, in the class of all measurable sets K in $\mathbf{R}^n$ of volume one, the integral $\int_K |x|^2\, dx$ is minimized for the normalized Euclidean ball B_n with center at the origin. Therefore, for isotropic K, we always have $L_K \geq L_{B_n}$ which leads to the optimal dimension-free lower bound

$$L_K \geq \frac{1}{\sqrt{2\pi e}}. \tag{2.2}$$

More generally, in the class of all probability densities q on $\mathbf{R}^n$ attaining maximum at the origin, the quantity $q^2(0) \int |x|^2 q(x)\, dx$ is minimized for the indicator function of B_n. This property was observed by D. Hensley [H] who assumed additionally that q is log-concave and symmetric, and later K. Ball [Ba] gave a shorter argument not using log-concavity and symmetry. In the one-dimensional case, the property reads as

$$q(0) \bigg(\int_{\mathbf{R}} t^2 q(t)\, dt \bigg)^{1/2} \geq \frac{1}{2\sqrt{3}}. \tag{2.3}$$

Remark 2.2. The inequality (2.1) is a particular case of the following theorem due to S. Karlin, F. Proschan, and R.E. Barlow [K-P-B]: Given a positive random variable ξ with a log-concave density on $(0, +\infty)$, for all real $s > 1$

$$\mathbf{E}\, \xi^s \leq \Gamma(s+1)\, (\mathbf{E}\, \xi)^s.$$

Equality is achieved if and only if ξ has an exponential distribution, that is, when $\mathrm{Prob}\{\xi > t\} = e^{-\lambda t}$, $t > 0$, for some parameter $\lambda > 0$.

Proposition 2.2. *For every hyperspace H in $\mathbf{R}^n$,*

$$\mathrm{vol}_{n-1}(K \cap H) \geq \frac{1}{\sqrt{6}}.$$

Moreover, if K is invariant under permutations of coordinates, then every section $K_j = K \cap \{x_j = 0\}$, $1 \leq j \leq n$, satisfies $\mathrm{vol}_{n-1}(K_j) \geq 1$.

Proof. If $H = \{x \in \mathbf{R}^n : \langle \theta, x\rangle = 0\}$, $|\theta| = 1$, apply (2.3) to the density $q(t)$ of the linear function $x \to \langle \theta, x\rangle$ over K: then we get

$$\mathrm{vol}_{n-1}(K \cap H)\, L_K \geq \frac{1}{2\sqrt{3}}.$$

This inequality holds true for any symmetric isotropic convex set K of volume one. In our specific case, it remains to apply Proposition 2.1.

For the second statement, given a non-empty set $\pi \subset \{1, \dots, n\}$, denote by K_π^+ the section of K by the $(n - |\pi|)$-dimensional subspace $\{x : x_j = 0,\ \text{for all}\ j \in \pi\}$. Write the Steiner decomposition

$$\mathrm{vol}_n\left(K^+ + r[0,1]^n\right) = \sum_{k=0}^{n} a_k(K^+)\, r^k, \quad r > 0,$$

where $a_k = \sum_{|\pi|=k} \mathrm{vol_{n-k}}(K_\pi^+)$ with the convention that $a_0 = \mathrm{vol}_n(K^+) = 1$. By the Brunn-Minkowski inequality, $\mathrm{vol}_n\left(K^+ + r\,[0,1]^n\right) \geq (1+r)^n$, so the coefficient $a_1(K^+)$ in front of r should satisfy $a_1 \geq n$. That is,

$$\sum_{j=1}^{n} \mathrm{vol}_{n-1}(K_j^+) \geq n,$$

where $K_j^+ = K^+ \cap \{x_j = 0\}$. Since all these $(n-1)$-dimensional volumes are equal to each other, and $\mathrm{vol}_{n-1}(K_j) = \mathrm{vol}_{n-1}(K_j^+)$, the conclusion follows.

Proposition 2.3. *For all* $\alpha_1, \dots, \alpha_n \geq 0$,

$$\mathrm{vol}_n\{x \in K^+ : x_1 \geq \alpha_1, \dots, x_n \geq \alpha_n\} \leq e^{-c\,(\alpha_1 + \ldots + \alpha_n)}$$

with $c = 1/\sqrt{6}$. *If* K *is invariant under permutations of coordinates, one may take* $c = 1$.

Proof. The function $u(\alpha_1, \dots, \alpha_n) = \mathrm{vol}_n\{x \in K^+ : x_1 \geq \alpha_1, \dots, x_n \geq \alpha_n\}$ is log-concave on $\mathbf{R}_+^n$, $u(0) = 1$, and

$$\left.\frac{\partial u(\alpha)}{\partial \alpha_j}\right|_{\alpha=0} = -\mathrm{vol}_{n-1}(K_j) \leq -c,$$

according to Proposition 2.2. These properties easily imply the desired inequality.

Actually, Proposition 2.3 can be sharpened by applying the Brunn-Minkowski inequality in its full volume. The latter implies that the function $u^{1/n}$ is concave on K^+ which is a slightly stronger property than just log-concavity. Hence, with the same argument, we have the inequality

$$\mathrm{vol}_n^{1/n}\{x \in K^+ : x_1 \geq \alpha_1, \dots, x_n \geq \alpha_n\} \leq 1 - \frac{c\,(\alpha_1 + \ldots + \alpha_n)}{n}$$

holding true for all $(\alpha_1, \dots, \alpha_n) \in K^+$ with $c = 1/\sqrt{6}$. Since the right hand side of this inequality must be non-negative, an immediate consequence of such a refinement is:

Proposition 2.4. *For all* $(x_1, \dots, x_n) \in K^+$,

$$x_1 + \ldots + x_n \le \sqrt{6}\, n.$$

Equivalently, for all $(x_1, \dots, x_n) \in K$, $|x_1| + \ldots + |x_n| \le \frac{\sqrt{6}}{2}\, n$.

Thus, the normalized ℓ^1-ball in $\mathbf{R}^n$ is the largest set within the class of all K's which we consider (up to a universal enlarging factor). One may wonder therefore whether or not it is true that the cube would be the smallest one. The question turns out simple as one can see from the proof of the following:

Proposition 2.5. *The set* K *contains the cube* $[-\frac{1}{\sqrt{2}}\, L_K, \frac{1}{\sqrt{2}}\, L_K]^n$ *which in turn contains* $[-\frac{1}{2\sqrt{\pi e}}, \frac{1}{2\sqrt{\pi e}}]^n$.

Proof. The baricenter $v = \mathrm{bar}(K^+)$ must belong to K^+, so K^+ contains parallepiped $\prod_{j=1}^n [0, v_j]$ with $v_j = \int_{K^+} x_j\, dx$. Hence the first statement immediately follows from the Khinchine-type inequality (2.1). The second one is based on the lower bound (2.2).

3 Log-Concave Measures

Here we extend Propositions 2.1–2.3 to log-concave measures. Let μ be a probability measure on $\mathbf{R}^n$ with a log-concave density $p(x)$, $x \in \mathbf{R}^n$, such that

a) $p(0) = 1$;

b) $p(\pm x_1, \dots, \pm x_n)$ does not depend on the choice of signs;

c) $\int x_j^2\, d\mu(x) = \int x_j^2\, p(x)\, dx = L_\mu^2$ does not depend on $j = 1, \dots, n$.

The case of the indicator density $p(x) = 1_K(x)$ reduces to the previous section. As in the body case, we associate with μ its squeezed restriction μ^+ to the positive octant $\mathbf{R}^n_+$: this measure has density

$$p^+(x) = p\left(\frac{1}{2}\, x\right), \quad x \in \mathbf{R}^n_+.$$

If $x = (x_1, \dots, x_n)$ is distributed according to μ, then the vector $(2|x_1|, \dots, 2|x_n|)$ is distributed according to μ^+. The function p^+ is log-concave, is non-increasing in each coordinate, and satisfies

$$\int_{\mathbf{R}^n_+} x_j^2\, d\mu^+(x) = 4L_\mu^2, \quad 1 \le j \le n.$$

Proposition 3.1. *$L_\mu \le C$, for some absolute C.*

Proof. Since p^+ is non-increasing, for every $x \in \mathbf{R}_+^n$,

$$1 \ge \int_0^{x_1} \dots \int_0^{x_n} p^+(y)\,dy \ge p^+(x) \int_0^{x_1} \dots \int_0^{x_n} dy = p^+(x) \prod_{j=1}^n x_j.$$

Hence,

$$u(x) \equiv -\log p^+(x) \ge \log \prod_{j=1}^n x_j \equiv v(x).$$

Note that u is convex, while v is a concave function. Therefore, there must exist an affine function ℓ such that $u(x) \ge \ell(x) \ge v(x)$, for all $x \in \mathbf{R}_+^n$. This function can be chosen to be tangent to v at some point $a = (a_1, \dots, a_n)$ with positive coordinates. That is, we may take

$$\ell(x) = v(a) + \big\langle \nabla v(a), x - a \big\rangle = \log \prod_{j=1}^n a_j + \sum_{j=1}^n \frac{x_j - a_j}{a_j}.$$

Setting $\lambda_j = \frac{1}{a_j}$, the inequality $u(x) \ge \ell(x)$ becomes

$$p^+(x) \le e^n \prod_{j=1}^n \lambda_j\, e^{-\lambda_j x_j}, \quad x \in \mathbf{R}_+^n.$$

In particular, since $p^+(0) = 1$, we have $\prod_{j=1}^n \lambda_j \ge e^{-n}$. Hence,

$$\int_{\mathbf{R}_+^n} \prod_{j=1}^n x_j\; p^+(x)\,dx \le \int_{\mathbf{R}_+^n} \prod_{j=1}^n x_j \Big(e^n \prod_{j=1}^n \lambda_j\, e^{-\lambda_j x_j} \Big) dx = e^n \prod_{j=1}^n \frac{1}{\lambda_j} \le e^{2n}.$$

On the other hand, with respect to μ^+,

$$\Big\| \prod_{j=1}^n x_j \Big\|_1 \ge \Big\| \prod_{j=1}^n x_j \Big\|_0 = \prod_{j=1}^n \|x_j\|_0 \ge c^n \prod_{j=1}^n \|x_j\|_2 = (2c)^n\, L_\mu^n,$$

where we have used a Khinchine-type inequality $\|g\|_0 = \lim_{p \to 0^+} \|g\|_p \ge c\,\|g\|_2$ for linear functions g with respect to log-concave measures (which is actually valid for any norm, cf. [L]). Proposition 3.1 follows with $C = e^2/(2c)$.

Proposition 3.2. *For every hyperspace H in $\mathbf{R}^n$,*

$$\int_H p(x)\,dx \ge \frac{1}{e\sqrt{6}}.$$

If p is invariant under permutations of coordinates, then $\int_{\{x_j = 0\}} p(x)\,dx \ge \frac{1}{e}$, for every $1 \le j \le n$.

There is a way to prove this statement without appealing to Proposition 3.1. In turn, starting from Proposition 3.2, one can easily obtain Proposition 3.1 with $C = e\sqrt{3}$. Indeed, the reverse one-dimensional Hensley inequality (for the class of all symmetric log-concave probability densities q on the line, cf. [H], Lemma 4) asserts that

$$q(0) \left(\int_{\mathbf{R}} t^2 \, dx \right)^{1/2} \leq \frac{1}{\sqrt{2}} \tag{3.1}$$

(equality is achieved at $q(t) = e^{-2|t|}$). If we take any hyperspace $H = \{x \in \mathbf{R}^n : \langle \theta, x \rangle = 0\}$, $|\theta| = 1$, and apply this inequality to the density $q(t)$ of the distribution of the linear function $\langle \theta, x \rangle$ under the measure μ, then we arrive exactly at

$$\int_H p(x) \, dx \, L_\mu \leq \frac{1}{\sqrt{2}}.$$

Hence, the lower bound $\int_H p(x) \, dx \geq 1/(e\sqrt{6})$ would lead to $L_\mu \leq e\sqrt{3}$, while in the case where μ is invariant under permutations of coordinates we would similarly obtain the estimate $L_\mu \leq e/\sqrt{2}$.

Proposition 3.2 will be derived from a more general:

Lemma 3.1. *For any log-concave probability density p on $\mathbf{R}^n$ such that $p(0) = 1$ and $p(\pm x_1, \dots, \pm x_n)$ does not depend on the choice of signs,*

$$\prod_{j=1}^n \int_{\{x_j = 0\}} p(x) \, dx \geq e^{-n}. \tag{3.2}$$

It is interesting that the constant $1/e$ appearing on the right is asymptotically optimal. Indeed, for the density

$$p(x) = \exp\Big\{ -2n!^{1/n} \max_{j \leq n} |x_j| \Big\},$$

for every $j \leq n$, we have $\int_{\{x_j=0\}} p(x) \, dx = \frac{n!^{1/n}}{n} \to \frac{1}{e}$, as $n \to \infty$.

As in this example, when a density p is invariant under permutations of coordinates, all $(n-1)$-dimensional integrals $\int_{\{x_j=0\}} p(x) \, dx$ coincide, so, by (3.2), these integrals must be greater or equal to $1/e$. In the general case, we may only conclude that $\max_j \int_{\{x_j=0\}} p(x) \, dx \geq 1/e$. On the other hand, the combination of the two Hensley's inequalities (2.3) and (3.1) immediately implies that, for any symmetric log-concave isotropic density p on $\mathbf{R}^n$ and for any two hyperspaces H_1, H_2, we have $\int_{H_1} p(x) \, dx \leq \sqrt{6} \int_{H_2} p(x) \, dx$. Hence,

$$\min_H \int_H p(x) \, dx \geq \frac{1}{\sqrt{6}} \max_j \int_{\{x_j=0\}} p(x) \, dx \geq \frac{1}{e\sqrt{6}}.$$

Thus, Lemma 3.1 implies Proposition 3.2.

Proof of Lemma 3.1. Given a measurable set A in $\mathbf{R}^n$, an inequality due to L. H. Loomis and H. Whitney asserts ([L-W], [B-Z]) that

$$\prod_{j=1}^n \mathrm{vol}_{n-1}(A_j) \geq \mathrm{vol}_n(A)^{n-1},$$

where A_j is the projection of A to the hyperspace $x_j = 0$. As a matter of fact, being applied to $A = K$, the above yields yet another proof of the second part of Proposition 2.2.

Loomis-Whitney's inequality admits a certain functional formulation. Namely, given a measurable function $g \geq 0$ on $\mathbf{R}^n$, not identically zero, consider the family $A(t) = \{x : g(x) > t\}$, $t > 0$. Define on $\mathbf{R}^{n-1}$ the functions

$$g_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) = \sup_{x_j} g_j(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_n)$$

together with $A_j(t) = \{x : g_j(x) > t\}$, $t > 0$. Then $A_j(t)$ are projections of $A(t)$, so

$$\mathrm{vol}_n\big(A(t)\big)^{n-1} \leq \prod_{j=1}^n \mathrm{vol}_{n-1}\big(A_j(t)\big).$$

Put $\varphi_j(t) = \mathrm{vol}_{n-1}(A_j(t))$, $\varphi(t) = \mathrm{vol}_n(A(t))$. Raising the above to the power $1/n$, integrating over $t > 0$ and applying Hölder's inequality, we get

$$\int_0^{+\infty} \varphi(t)^{(n-1)/n}\, dt \leq \int_0^{+\infty} \prod_{j=1}^n \varphi_j(t)^{1/n}\, dt \leq \prod_{j=1}^n \bigg(\int_0^{+\infty} \varphi_j(t)\, dt\bigg)^{1/n}$$
$$= \bigg(\prod_{j=1}^n \int_{\mathbf{R}^{n-1}} g_j(x)\, dx\bigg)^{1/n}.$$

In order to bound from below the first integral, we use the property that $\varphi(t)$ is non-increasing in $t > 0$. For such functions, for all $\alpha \in (0, 1]$, there is a simple inequality (cf. [B-Z])

$$\bigg(\int_0^{+\infty} \varphi(t)^\alpha\, dt\bigg)^{1/\alpha} \geq \int_0^{+\infty} \varphi(t^\alpha)\, dt.$$

But the right hand side is exactly $\int_{\mathbf{R}^n} g(x)^{1/\alpha}\, dx$, and for $\alpha = \frac{n-1}{n}$, we thus get

$$\prod_{j=1}^n \int_{\mathbf{R}^{n-1}} g_j(x)\, dx \geq \bigg(\int_{\mathbf{R}^n} g(x)^{n/(n-1)}\, dx\bigg)^{n-1}.$$

This is the desired functional form yielding the original inequality on indicator functions $g = 1_A$. For $g = p$, the supremum in the definition of g_j is attained at $x_j = 0$, and the functional inequality becomes

$$\prod_{j=1}^{n} \int_{\{x_j=0\}} p(x)\,dx \geq \left(\int_{\mathbf{R}^n} p(x)^{n/(n-1)}\,dx \right)^{n-1}.$$

The right hand side can further be estimated using the log-concavity of p. Namely, since $p(0) = 1$, for every $t \in (0,1)$ and $x \in \mathbf{R}^n$, we have $p(tx)^{1/t} \geq p(x)$. Integrating over x, we get $\int_{\mathbf{R}^n} p(x)^{1/t}\,dx \geq t^n$ which for $t = \frac{n-1}{n}$ gives

$$\int_{\mathbf{R}^n} p(x)^{n/(n-1)}\,dx \geq \left(\frac{n-1}{n}\right)^n, \quad n \geq 2.$$

It remains to note that $\left(\frac{n-1}{n}\right)^{n(n-1)} \geq e^{-n}$.

Lemma 3.1 follows. As a consequence, we get an analogue of Proposition 2.3:

Proposition 3.3. *For all* $\alpha_1, \dots, \alpha_n \geq 0$,

$$\mu^+\{x \in \mathbf{R}_+^n : x_1 \geq \alpha_1, \dots, x_n \geq \alpha_n\} \leq e^{-c\,(\alpha_1 + \dots + \alpha_n)}$$

with $c = \frac{1}{e\sqrt{6}}$. *If* μ *is invariant under permutations of coordinates, one may take* $c = 1/e$.

4 Decreasing Rearrangement

For any vector $x = (x_1, \dots, x_n)$ in $\mathbf{R}^n$, its coordinates can be written in the decreasing order,

$$X_1 \geq X_2 \geq \dots \geq X_n.$$

In particular, $X_1 = \max_j x_j$, $X_n = \min_j x_j$. When x is observed as a random vector with uniform distribution in K^+ or more generally with distribution μ^+, the distribution of the random vector $(X_1, \dots, X_n)$ can be studied on the basis of Propositions 2.3 and 3.3, respectively. In particular, we have:

Proposition 4.1. *For any* $\alpha \geq 0$, $1 \leq k \leq n$,

$$\mu^+\{x \in \mathbf{R}_+^n : X_k \geq \alpha\} \leq C_n^k\, e^{-c\,k\alpha},$$

where $c > 0$ *is a numerical constant.*

One may always take $c = 1/(e\sqrt{6}\,)$ but the constant can be improved for special situations. For example, $c = 1/e$, when μ^+ is invariant under permutations of coordinates, and moreover $c = 1$ when μ^+ is uniform distribution on K^+ which is invariant under permutations of coordinates.

We denote by C_n^k the usual combinatorial coefficients $\frac{n!}{k!(n-k)!}$.

Proof. Since

$$\{x \in \mathbf{R}_+^n : X_k \geq \alpha\} = \cup_{n \geq j_1 > \ldots > j_k \geq 1} \{x \in \mathbf{R}_+^n : x_{j_1} \geq \alpha, \ldots, x_{j_k} \geq \alpha\},$$

we get

$$\mu^+\{X_k \geq \alpha\} \;\leq \sum_{n \geq j_1 > \ldots > j_k \geq 1} \mu^+\{x_{j_1} \geq \alpha, \ldots, x_{j_k} \geq \alpha\} \;\leq\; C_n^k \, e^{-c\,k\alpha},$$

where we applied Proposition 3.3 (or, respectively, Proposition 2.3) on the last step.

The combinatorial argument easily extends to yield a more general:

Proposition 4.2. *For any collection of indices* $1 \leq k_1 < \ldots < k_r \leq n$, *and for all* $\alpha_1, \ldots, \alpha_r \geq 0$,

$$\mu^+\{X_{k_1} \geq \alpha_1, \ldots, X_{k_r} \geq \alpha_r\} \leq \frac{n! \, e^{-c\,(k_1\alpha_1 + (k_2 - k_1)\alpha_2 \ldots + (k_r - k_{r-1})\alpha_r)}}{k_1!(k_2 - k_1)! \ldots (k_r - k_{r-1})!(n - k_r)!},$$

where $c > 0$ *is a numerical constant.*

Let us now illustrate one of the possible applications to large deviations, say, for ℓ^1-norm $\|x\|_1 = \sum_{k=1}^n |x_k|$ under the measure μ. For all numbers $\alpha_1, \ldots, \alpha_n \geq 0$,

$$\mu\left\{\|x\|_1 \geq \sum_{k=1}^n \alpha_k\right\} = \mu^+\left\{\sum_{k=1}^n x_k \geq 2\sum_{k=1}^n \alpha_k\right\} \;=\; \mu^+\left\{\sum_{k=1}^n X_k \geq 2\sum_{k=1}^n \alpha_k\right\}$$

$$\leq \sum_{k=1}^n \mu^+\{X_k \geq 2\,\alpha_k\} \;\leq\; \sum_{k=1}^n C_n^k \, e^{-2c\,k\,\alpha_k}$$

where we applied Proposition 4.1 on the last step. Using $C_n^k \leq \left(\frac{ne}{k}\right)^k$, we thus get

$$\mu\left\{c\,\|x\|_1 \geq \sum_{k=1}^n \alpha_k\right\} \leq \sum_{k=1}^n e^{-k\,\left(2\alpha_k - \log\frac{ne}{k}\right)}.$$

Now, take $\alpha_k = \frac{1}{2}\log\frac{ne}{k} + t\,\frac{n}{k(\log n + 1)}$ which is almost an optimal choice. Then, $\sum_{k=1}^n \alpha_k \leq n(1+t)$, and we arrive at:

Proposition 4.3. *For any* $t \geq 0$,

$$\mu\left\{\frac{c\,\|x\|_1}{n} \geq 1 + t\right\} \;\leq\; n\,\exp\left\{-2t\,\frac{n}{\log n + 1}\right\}.$$

The right hand side converges to zero for any fixed $t > 0$. In particular, for large n, we have $\|x\|_1 \leq 2n/c$ with μ-probability almost one. In probabilistic language, this means that the random variables $\|x\|_1/n$ are stochastically bounded as $n \to \infty$. Since $L^1(\mu)$-norm of $\|x\|_1/n$ is about 1, this property cannot be deduced from the usual exponential bound for norms under log-concave measures (cf. [Bo]).

5 Euclidean Norm. Proof of Theorem 1.2

As in the proof of Proposition 4.3, for all $\alpha_1, \dots, \alpha_n \geq 0$, we similarly obtain that

$$\mu\left\{|x|^2 \geq \sum_{k=1}^n \alpha_k^2\right\} = \mu^+\left\{\sum_{k=1}^n X_k^2 \geq 4\sum_{k=1}^n \alpha_k^2\right\}$$
$$\leq \sum_{k=1}^n \mu^+\{X_k \geq 2\,\alpha_k\} \ \leq \ \sum_{k=1}^n C_n^k\, e^{-2c\,k\,\alpha_k}$$

where again we applied Proposition 4.1 on the last step. Using $C_n^k \leq \left(\frac{ne}{k}\right)^k$, we thus get

$$\mu\left\{c^2\,|x|^2 \geq \sum_{k=1}^n \alpha_k^2\right\} \leq \sum_{k=1}^n e^{-k\,\left(2\alpha_k - \log\frac{ne}{k}\right)}.$$

Now, take $\alpha_k = \frac{1}{2}\log\frac{ne}{k} + t\,\frac{\sqrt{n}}{k}$. Then, $\sum_{k=1}^n \alpha_k^2 \leq 4nt^2$, for all $t \geq 2$, so

$$\mu\left\{\frac{c\,|x|}{\sqrt{n}} \geq 2t\right\} \ \leq \ n\,e^{-2t\sqrt{n}}.$$

In a more compact form:

Proposition 5.1. *For any $t \geq 4$,*

$$\mu\left\{x \in \mathbf{R}^n : \frac{c\,|x|}{\sqrt{n}} \geq t\right\} \ \leq \ e^{-\frac{1}{2}\,t\sqrt{n}}.$$

As in Proposition 4.1, we may take $c = 1/(e\sqrt{6}\,)$ in general, and $c = 1/\sqrt{6}$ in the body case. As explained in section 1, the above inequality implies:

Proposition 5.2. *For every number $C \geq 56$, in the interval $0 \leq t \leq Cn^{1/4}$,*

$$\mu\left\{x \in \mathbf{R}^n : \left|\frac{x_1 + \ldots + x_n}{\sqrt{n}}\right| \geq t\right\} \ \leq \ 2\,\exp\left\{-\frac{t^2}{8\,C^{4/3}}\right\}.$$

Indeed, applying Proposition 5.1 with $c = \frac{1}{7} < \frac{1}{e\sqrt{6}}$, we get

$$\mu\left\{\frac{1}{\sqrt{n}}\left|\sum_{j=1}^n x_j\right| \geq t\right\} = \mu \otimes \mathbf{P}_\varepsilon\left\{\frac{1}{\sqrt{n}}\left|\sum_{j=1}^n \varepsilon_j x_j\right| \geq t\right\}$$
$$\leq \int e^{-nt^2/(2|x|^2)}\,d\mu(x) \ = \ \int_{|x| \leq t_0\sqrt{n}} + \int_{|x| \geq t_0\sqrt{n}}$$
$$\leq e^{-t^2/(2t_0^2)} + e^{-\frac{1}{14}\,t_0\sqrt{n}},$$

for every t_0 provided that $ct_0 \geq 4$, that is, $t_0 \geq 28$. By the assumption on t, the last term is bounded by $e^{-t_0\,t^2/(14\,C^2)}$. It remains to take (the optimal) $t_0 = (7C^2)^{1/3}$.

6 Theorem 1.1 for Log-Concave Measures

In order to involve the region $t \geq Cn^{1/4}$ in Proposition 5.2, an extra condition on the measure μ is required. One important property distinguishing the case where μ is the uniform distribution on K from the general measure case is indicated in Proposition 2.4: for all $x \in K$,

$$|x_1| + \ldots + |x_n| \leq An \tag{6.1}$$

with $A = \sqrt{6}/2$. It is therefore natural to assume that the measure μ is supported on a convex set satisfying (6.1) for some $A = A(\mu)$. In this case Theorem 1.1 admits a corresponding extension:

Proposition 6.1. $\|f\|_{L_{\psi_2}(\mu)} \leq C\sqrt{A(\mu)}$, *where C is a numerical constant.*

Note that in terms of the linear functional

$$f(x) = \frac{x_1 + \ldots + x_n}{\sqrt{n}}$$

the quantity $A(\mu)$ is described as $1/\sqrt{n}\,\|f\|_{L_\infty(\mu)}$. Thus, Proposition 6.1 relates L_{ψ_2}-norm to L_∞-norm of f via $\|f\|_{L_{\psi_2}(\mu)} \leq C/\sqrt{n}\,\sqrt{\|f\|_{L_\infty(\mu)}}$. This inequality is not linear in f which is due to the basic assumption $p(0) = 1$ on the density p of μ. Without this condition, Proposition 6.1 can be formulated as follows:

Corollary 6.1. *Let μ be a probability measure on $\mathbf{R}^n$ with a log-concave density p such that, for all $x \in \mathbf{R}^n$, $p(\pm x_1, \ldots, \pm x_n)$ does not depend on the choice of signs, and $\int_{\mathbf{R}^n} x_j^2\, p(x)\, dx$ does not depend on $j = 1, \ldots, n$. Then, for some universal C,*

$$\|f\|^2_{L_{\psi_2}(\mu)} \leq \frac{C}{\sqrt{n}}\, \|f\|_{L_2(\mu)}\, \|f\|_{L_\infty(\mu)}.$$

Let us return to the original assumption $p(0) = 1$. Then $A(\mu)$ is always separated from zero. Indeed, since the density $p(x)$ is bounded by 1, we have

$$\begin{aligned} 1 &= \int_{|x_1|+\ldots+|x_n| \leq An} p(x)\, dx \leq \mathrm{vol}_n\{x \in \mathbf{R}^n : |x_1| + \ldots + |x_n| \leq An\} \\ &= \frac{(2An)^n}{n!}. \end{aligned}$$

Hence, $A \geq \frac{n!^{1/n}}{2n} \geq \frac{1}{2e}$.

While the first applications are based upon Proposition 4.1, the proof of Proposition 6.1 uses a more general Proposition 4.2. The estimate given in it can be simplified as follows: using a general bound $m! \geq \left(\frac{m}{e}\right)^m$ and the fact that the function $x \to \left(\frac{ne}{x}\right)^x$ increases in $0 < x \leq n$, we get

$$\frac{n(n-1)\dots(n-k_r+1)}{k_1!(k_2-k_1)!\dots(k_r-k_{r-1})!} \leq \prod_{j=1}^{r}\left(\frac{ne}{k_j-k_{j-1}}\right)^{k_j-k_{j-1}} \leq \prod_{j=1}^{r}\left(\frac{ne}{k_j}\right)^{k_j}$$

with the convention that $k_0 = 0$ on the middle step. Hence, for all $\alpha_1, \dots, \alpha_r \geq 0$,

$$\mu^+\{X_{k_1} \geq \alpha_1, \dots, X_{k_r} \geq \alpha_r\} \leq \\ \prod_{j=1}^{r}\left(\frac{ne}{k_j}\right)^{k_j} e^{-c\,(k_1\alpha_1+(k_2-k_1)\alpha_2 \dots +(k_r-k_{r-1})\alpha_r)}.$$

From now on, the indices k_j will be assumed to be the powers of 2. Thus let $\ell = [\log_2 n]$ (the integer part), and let S be any non-empty subset of $\{0, 1, \dots, \ell\}$. From the previous inequality, for any collection $\alpha_k \geq 0$ indexed by $k \in S$,

$$\mu^+\left\{X_{2^k} \geq \alpha_k, \text{ for all } k \in S\right\} \;\leq\; \prod_{k\in S}\left(\frac{ne}{2^k}\right)^{2^k} \exp\Big\{-c\sum_{k\in S}2^{k-1}\alpha_k\Big\}.$$

The choice $\alpha_k = \beta_k + \frac{2}{c}\log\frac{ne}{2^k}$ leads to:

Lemma 6.1. *For any non-empty subset S of $\{0, 1, \dots, \ell\}$ and any collection $\beta = (\beta_k)_{k\in S}$ of non-negative numbers,*

$$\mu^+\Big\{X_{2^k} \geq \beta_k + \frac{2}{c}\log\frac{ne}{2^k}, \text{ for all } k \in S\Big\} \;\leq\; \exp\Big\{-c\sum_{k\in S}2^{k-1}\beta_k\Big\}.$$

As before, one may take $c = 1/(e\sqrt{6})$. In view of the assumption (6.1), the measure μ^+ is supported by

$$x_1 + \dots + x_n \leq 2An$$

so, only $\beta_k < 2An$ can be of interest in Lemma 6.1. Assume moreover that each β_k also represents a power of 2. The couples (S, β) with these properties will be called blocks, and we say that a vector $x \in \mathbf{R}^n_+$ is controlled by a block (S, β) if

$$X_{2^k} \geq \beta_k + \frac{2}{c}\log\frac{ne}{2^k}, \quad \text{for all } k \in S.$$

Lemma 6.2. *The total number of blocks does not exceed, $e^{2\log 2n\log(2\log 4An)}$.*

Indeed, given a non-empty $S \subset \{0, 1, \dots, \ell\}$, the number of admissible functions β on S is equal to $[\log_2 2An]^{|S|}$. Hence, the number of all blocks is equal to

$$\sum_S[\log_2 2An]^{|S|} = \sum_{r=1}^{\ell+1}C^r_{\ell+1}[\log_2 2An]^r = \big(1+[\log_2 2An]\big)^{[\log_2 n]+1} - 1$$

from which the desired bound easily follows.

Combining Lemma 6.1 with Lemma 6.2 and using $c = 1/(e\sqrt{6}) > 1/7$, we thus obtain that

$$\mu^+\Big\{x \in \mathbf{R}^n_+ : x \text{ is controlled by a block } (S,\beta) \text{ with } \sum_{k\in S} 2^{k-1}\beta_k \geq \frac{1}{8}\,t\sqrt{n}\Big\}$$
$$\leq e^{2\log 2n \log(2\log 4An)}\, e^{-\frac{1}{8\cdot 7}\,t\sqrt{n}}. \tag{6.2}$$

Lemma 6.3. *Given $t > 0$, assume that a vector $x \in \mathbf{R}^n_+$ is not controlled by any block (S,β) with $\sum_{k\in S} 2^{k-1}\beta_k \geq \frac{1}{8}\,t\sqrt{n}$. Then, with some absolute constant $B > 0$,*

$$\mathbf{P}_\varepsilon\left\{\left|\frac{\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n}{\sqrt{n}}\right| \geq t\right\} \leq 2\,e^{-t^2/B}.$$

Proof. It is also possible that x is not controlled by any block (S,β) at all: by the very definition, this holds if and only if

$$X_{2^k} < 1 + \frac{2}{c}\,\log\frac{ne}{2^k}, \quad \text{for all } 0 \leq k \leq \ell.$$

But then

$$|x|^2 = \sum_{j=1}^n X_j^2 \leq \sum_{k=0}^{\ell} X_{2^k}^2\, 2^k < \sum_{k=0}^{\ell}\left(1 + \frac{2}{c}\,\log\frac{ne}{2^k}\right)^2 2^k \leq Bn,$$

for some absolute constant B. Therefore, for all $t > 0$,

$$\mathbf{P}_\varepsilon\left\{\left|\frac{\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n}{\sqrt{n}}\right| \geq t\right\} \leq 2\,e^{-nt^2/2|x|^2} \leq 2\,e^{-t^2/2B},$$

and the statement follows.

In the other case, there is a maximal block controlling the given vector x. Namely, introduce (the canonical) set

$$S = \left\{k = 0,1,\ldots,\ell : X_{2^k} \geq 1 + \frac{2}{c}\,\log\frac{ne}{2^k}\right\},$$

and for each $k \in S$, denote by β_k the maximal power of 2 not exceeding $X_{2^k} - \frac{2}{c}\,\log\frac{ne}{2^k}$. In particular,

$$\beta_k \leq X_{2^k} - \frac{2}{c}\,\log\frac{ne}{2^k} < 2\beta_k, \tag{6.3}$$

and, by the assumption of the lemma,

$$\sum_{k\in S} 2^{k-1}\beta_k < \frac{1}{8}\,t\sqrt{n}. \tag{6.4}$$

Define a new vector $(Y_j)_{1\le j\le n}$ approximating $(X_j)_{1\le j\le n}$ in a certain sense. First put

$$\alpha_k = \left(X_{2^k} - \left(1 + \frac{2}{c}\,\log\frac{ne}{2^k}\right)\right)^+, \quad 0 \le k \le \ell,$$

so that $\alpha_k = 0$ outside S and $0 \le \alpha_k \le 2\beta_k - 1 < 2\beta_k$, for all $k \in S$, according to (6.3). Let $Y_j = (X_j - \alpha_k)^+$, for $2^k \le j < 2^{k+1}$ $(0 \le k \le \ell)$. Then, clearly $0 \le Y_j \le X_j \le Y_j + \alpha_k$, and by (6.4),

$$\sum_{j=1}^n X_j - Y_j \le \sum_{k=0}^{\ell} 2^k \alpha_k = \sum_{k\in S} 2^k \alpha_k \le \sum_{k\in S} 2^{k+1}\beta_k < \frac{1}{2}\,t\sqrt{n}.$$

Hence,

$$\mathbf{P}_\varepsilon\Big\{\frac{1}{\sqrt{n}}\Big|\sum_{j=1}^n \varepsilon_j x_j\Big| \ge t\Big\} = \mathbf{P}_\varepsilon\Big\{\frac{1}{\sqrt{n}}\Big|\sum_{j=1}^n \varepsilon_j X_j\Big| \ge t\Big\}$$
$$\le \mathbf{P}_\varepsilon\Big\{\frac{1}{\sqrt{n}}\Big|\sum_{j=1}^n \varepsilon_j Y_j\Big| \ge \frac{t}{2}\Big\}.$$

It remains to observe that, for $2^k \le j < 2^{k+1}$, we have $Y_j \le Y_{2^k} \le 1 + \frac{2}{c}\,\log\frac{ne}{2^k}$, so $\sum_{j=1}^n Y_j^2 \le \sum_{k=0}^{\ell}\big(1 + \frac{2}{c}\,\log\frac{ne}{2^k}\big)^2\, 2^k \le Bn$. Lemma 6.3 follows.

Proof of Proposition 6.1. We need to get a subgaussian bound of the form $\mu\{|f| \ge t\} \le c_1\, e^{-c_2 t^2/A}$, for some absolute $c_1, c_2 > 0$. By the assumption (6.1) on the support of μ, we may assume $t \le A\sqrt{n}$.

Put $C = (\sigma A)^{3/4}$ with a positive universal constant σ to be determined later on. Since necessarily $A \ge 1/(2e)$, we assume $\big(\frac{\sigma}{2e}\big)^{3/4} \ge 56$ so that to apply Proposition 5.2 in the interval $0 \le t \le Cn^{1/4}$: it then gives

$$\mu\big\{x \in \mathbf{R}^n : |f(x)| \ge t\big\} \le 2e^{-t^2/(8\sigma A)}.$$

The right hand side is of the desired order both in t and A in that interval.

Now, let $t \ge Cn^{1/4}$. Define $\Omega_0(t)$ to be the collection of all vectors $x \in \mathbf{R}^n_+$ which are controlled by a block (S,β) with $\sum_{k\in S} 2^{k-1}\beta_k \ge \frac{1}{8}\,t\sqrt{n}$. Let $\Omega_1(t) = \mathbf{R}^n_+ \setminus \Omega_0(t)$. In terms of $f(x,\varepsilon) = \frac{\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n}{\sqrt{n}}$, we may write

$$\mu\{|f| > t\} = \mu^+ \otimes \mathbf{P}_\varepsilon\big\{(x,\varepsilon) : |f(x,\varepsilon)| > 2t\big\}$$
$$= \int_{\Omega_0} \mathbf{P}_\varepsilon\big\{|f(x,\varepsilon)| > 2t\big\}\, d\mu^+(x) + \int_{\Omega_1} \mathbf{P}_\varepsilon\big\{|f(x,\varepsilon)| > 2t\big\}\, d\mu^+(x).$$

The second integral does not exceed $2e^{-t^2/B}$ with some numerical B (Lemma 6.3). The first integral can be bounded, according to (6.2), by

$$\mu^+\big(\Omega_0(t)\big) \le e^{-\frac{1}{56}\,t\sqrt{n} + \Delta_n(A)},$$

where $\Delta_n(A) = 2\log(2n)\log(2\log(4An))$. Thus, for the values $Cn^{1/4} \leq t \leq A\sqrt{n}$, it suffices to show that

$$e^{-\frac{1}{56}\,t\sqrt{n}+\Delta_n(A)} \leq e^{-t^2/(112\,A)}$$

(note that if $A\sqrt{n} < Cn^{1/4}$, we are done). Equivalently,

$$\frac{1}{112\,A}\,t^2 - \frac{1}{56}\,t\sqrt{n} + \Delta_n(A) \leq 0.$$

Since $t \leq A\sqrt{n}$, the above is implied by $\Delta_n(A) \leq \frac{1}{112}\,t\sqrt{n}$. In view of $t \geq Cn^{1/4} = (\sigma A)^{3/4}n^{1/4}$, the latter is equivalent to

$$\Delta_n(A) \leq \frac{1}{112}\,(\sigma A)^{3/4}n^{1/4}.$$

Clearly, if σ is sufficiently large, the above inequality holds true for all $A \geq \frac{1}{2e}$ and $n \geq 1$. Summarizing, we may write the following estimate for all $t > 0$:

$$\mu\{|f| > t\} \leq \max\left\{2e^{-t^2/(8\sigma A)},\ 2e^{-t^2/B} + e^{-t^2/(112\,A)}\right\}.$$

This gives the desired result.

References

[A] Alesker, S. (1995): ψ_2-estimate for the Euclidean norm on a convex body in isotropic position. Geom. Aspects Funct. Anal. (Israel 1992-1994), Oper. Theory Adv. Appl., **77**, 1–4

[Ba] Ball, K.M. (1988): Logarithmically concave functions and sections of convex bodies. Studia Math., **88**, 69–84

[Bo] Borell, C. (1974): Convex measures on locally convex spaces. Ark. Math., **12**, 239–252

[B-Z] Burago, Yu.D., Zalgaller, V.A. (1988): Geometric Inequalities. Springer-Verlag, Berlin

[H] Hensley, D. (1980): Slicing convex bodies – bounds for slice area in terms of the body's covariance. Proc. Amer. Math. Soc., **79(4)**, 619–625

[K-P-B] Karlin, S., Proschan, F., Barlow, R.E. (1961): Moment inequalities of Pólya frequency functions. Pacific J. Math., **11**, 1023–1033

[L] Latala, R. (1996): On the equivalence between geometric and arithmetic means for log-concave measures. In: Convex Geometric Analysis (Berkeley, CA, 1996), pp. 123–127; Math. Sci. Res. Inst. Publ., **34**, Cambridge Univ. Press, Cambridge, 1999

[L-W] Loomis, L.H., Whitney, H. (1949): An inequality related to the isoperimetric inequality. Bull. Amer. Math. Soc., **55**, 961–962

[M-P] Milman, V.D., Pajor, A. (1989): Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. Geom. Aspects of Funct. Anal. (1987–88), Lecture Notes in Math., **1376**, 64–104

[S-Z1] Schechtman, G., Zinn, J. (1990): On the volume of the intersection of two L_p^n balls. Proc. of the Amer. Math. Soc., **110**, 217–224

[S-Z2] Schechtman, G., Zinn, J. (2000): Concentration on the ℓ_p^n ball. GAFA, Israel Seminar 1996-2000, Lecture Notes in Math., Springer, **1745**, 247–256