Spectral Gap and Concentration for Some Spherically Symmetric Probability Measures*

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Summary. We study the spectral gap and a related concentration property for a family of spherically symmetric probability measures.

This note appeared in an attempt to answer the following question raised by V. Bogachev: How do we effectively estimate the spectral gap for the exponential measures μ on the Euclidean space $\mathbf{R}^n$ with densities of the form $\frac{d\mu(x)}{dx} = ae^{-b|x|}$?

By the spectral gap, we mean here the best constant $\lambda_1 = \lambda_1(\mu)$ in the Poincaré-type inequality

$$\lambda_1 \int_{\mathbf{R}^n} |u(x)|^2 \, d\mu(x) \leq \int_{\mathbf{R}^n} |\nabla u(x)|^2 \, d\mu(x) \tag{1}$$

with u being an arbitrary smooth (or, more generally, locally Lipschitz) function on $\mathbf{R}^n$ such that $\int u(x)\, d\mu(x) = 0$. Although it is often known that $\lambda_1 > 0$, in many problems of analysis and probability, one needs to know how the dimension n reflects on this constant. One important case, the canonical Gaussian measure $\mu = \gamma_n$, with density $(2\pi)^{-n/2}\, e^{-|x|^2/2}$, provides an example with a dimension-free spectral gap $\lambda_1 = 1$. This fact can already be used to recover a dimension-free concentration phenomenon in Gauss space.

To unite both the Gaussian and the exponential cases, we consider a spherically symmetric probability measure μ on $\mathbf{R}^n$ with density

$$\frac{d\mu(x)}{dx} = \rho(|x|), \quad x \in \mathbf{R}^n,$$

assuming that $\rho = \rho(t)$ is an arbitrary log-concave function on $(0, +\infty)$, that is, the function $\log \rho(t)$ is concave on its support interval. In order that μ be log-concave itself (cf. [Bor2] for a general theory of log-concave measures), ρ has also to be non-increasing in $t > 0$. However, this will not be required.

It is a matter of normalization, if we assume that μ satisfies

$$\int_{\mathbf{R}^n} \langle x, \theta \rangle^2 \, d\mu(x) = |\theta|^2, \quad \text{for all} \ \ \theta \in \mathbf{R}^n. \tag{2}$$

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As usual, $\langle \cdot, \cdot \rangle$ and $| \cdot |$ denote the scalar product and the Euclidean norm, respectively. Since μ is symmetrically invariant, this normalization condition may also be written as $\int x_1^2 \, d\mu(x) = 1$, or $\int |x|^2 \, d\mu(x) = n$. We prove:

Theorem 1. *Under* (2), *the optimal value of* λ_1 *in* (1) *satisfies* $\frac{1}{13} \leq \lambda_1 \leq 1$.

Returning to the exponential measure $d\mu(x) = a\, e^{-b|x|} dx$, $b > 0$, we thus obtain that λ_1 is of order b^2/n.

Using Theorem 1 and applying Gromov–Milmans's theorem on concentration under Poincaré-type inequalities, one may conclude that all the considered measures share a dimension-free concentration phenomenon:

Theorem 2. *Under* (2), *given a measurable set* A *in* $\mathbf{R}^n$ *of measure* $\mu(A) \geq \frac{1}{2}$, *for all* $h > 0$,

$$1 - \mu(A^h) \leq 2e^{-ch}, \tag{3}$$

where c *is a certain positive universal constant.*

Here, we use $A^h = \{x \in \mathbf{R}^n : \mathrm{dist}(A, x) < h\}$ to denote an h-neighborhood of A with respect to the Euclidean distance.

Note that, in polar coordinates, every spherically symmetric measure μ with density $\rho(|x|)$ represents a product measure, i.e., it may be viewed as the distribution of $\xi\theta$, where θ is a random vector uniformly distributed over the unit sphere S^{n-1}, and where $\xi > 0$ is an independent of θ random variable with distribution function

$$\mu\{|x| \leq t\} = n\omega_n \int_0^t s^{n-1} \rho(s) \, ds, \quad t > 0 \tag{4}$$

(ω_n is the volume of the unit ball in $\mathbf{R}^n$). For example, one can take $(\mathbf{R}^n, \mu)$ for the underlying probability space and put $\xi(x) = |x|$, $\theta(x) = \frac{x}{|x|}$. It is a classical fact that $\lambda_1(S^{n-1}) = n - 1$. To reach Theorems 1-2, our task will be therefore to estimate $\lambda_1(\xi)$ from below and to see in particular that the values of ξ are strongly concentrated around its mean $\mathbf{E}\xi$ which is of order $\sqrt{n}$. When ρ is log-concave, the density $q(t) = n\omega_n \, t^{n-1} \rho(t)$ of ξ is log-concave, as well. Of course, this observation is not yet enough to reach the desired statements, since it "forgets" about an important factor t^{n-1}. As a first step, we will need the following one-dimensional:

Lemma 1. *Given a positive integer* n, *if a random variable* $\xi > 0$ *has density* $q(t)$ *such that the function* $q(t)/t^{n-1}$ *is log-concave on* $(0, +\infty)$, *then*

$$\mathrm{Var}(\xi) \leq \frac{1}{n} \, (\mathbf{E}\xi)^2. \tag{5}$$

As usual, $\mathrm{Var}(\xi) = \mathbf{E}\xi^2 - (\mathbf{E}\xi)^2$ and $\mathbf{E}\xi$ denote the variance and the expectation of a random variable ξ.

For $\xi(x) = |x|$ as above, with distribution given by (4), in view of the normalization condition (2), we have $\mathbf{E}\xi^2 = n$, so the bound (5) yields a dimension-free inequality

$$\mathrm{Var}(\xi) \leq 1. \tag{6}$$

Lemma 1 represents a particular case of a theorem due to R.E. Barlow, A.W. Marshall, and F. Proshan (cf. [B-M-P], p. 384, and [Bor1]) which states the following: If a random variable $\eta > 0$ has a distribution with increasing hazard rate (in particular, if η has a log-concave density), then its normalized moments $\lambda_a = \frac{1}{\Gamma(a+1)}\,\mathbf{E}\eta^a$ satisfy a reverse Lyapunov's inequality

$$\lambda_a^{b-c}\lambda_c^{a-b} \leq \lambda_b^{a-c}, \qquad a \geq b \geq c \geq 1, \quad c \text{ integer}. \tag{7}$$

Indeed, putting $a = n+1$, $b = n$, $c = n-1$ $(n \geq 2)$, we get

$$\mathbf{E}\eta^{n+1}\,\mathbf{E}\eta^{n-1} \leq \left(1 + \frac{1}{n}\right)(\mathbf{E}\eta^n)^2. \tag{8}$$

If the random variable ξ has density $q(t) = t^{n-1}p(t)$ with p log-concave on $(0,+\infty)$, and η has density $p(t)/\int_0^{+\infty} p(t)\,dt$, the above inequality becomes $\mathbf{E}\xi^2 \leq (1 + \frac{1}{n})\,(\mathbf{E}\xi)^2$ which is exactly (5).

When $n = 1$, the latter is equivalent to the well-known Khinchine-type inequality $\mathbf{E}\eta^2 \leq 2\,(\mathbf{E}\eta)^2$. More generally, one has

$$\mathbf{E}\eta^a \leq \Gamma(a+1)\,(\mathbf{E}\eta)^a, \quad a \geq 1,$$

which is known to hold true in the class of all random variables $\eta > 0$ with log-concave densities. This fact cannot formally be deduced from (7) because of the assumption $c \geq 1$. It was obtained in 1961 by S. Karlin, F. Proshan, and R.E. Barlow [K-P-B] as an application of their study of the so-called totally positive functions (similar to [B-M-P] − with techniques and ideas going back to the work of I.J. Schoenberg [S]).

To make the proof of Theorem 1 more self-contained, we would like to include a different argument leading to the inequality (7) for a related function:

Lemma 2. *Given a log-concave random variable $\eta > 0$, the function $\lambda_a = \frac{1}{a^a}\,\mathbf{E}\eta^a$ is log-concave in $a > 0$. Equivalently, it satisfies* (7), *for all $a \geq b \geq c > 0$.*

Again putting $a = n+1$, $b = n$, $c = n-1$, we obtain $\mathbf{E}\eta^{n+1}\,\mathbf{E}\eta^{n-1} \leq C_n(\mathbf{E}\eta^n)^2$ with constant $C_n = \frac{(n+1)^{n+1}(n-1)^{n-1}}{n^{2n}}$ which is a little worse than that of (8). On the other hand, one can easily see that $C_n \leq 1 + \frac{1}{n} + \frac{1}{n^3}$, so, we get, for example, the constant $\frac{2}{n}$ in Lemma 1 (and this leads to the lower bound $\frac{1}{25}$ in Theorem 1).

Finally, it might also be worthwhile to mention here the following interesting immediate consequence of Lemmas 1-2. Given an integer $d \geq 1$ and an arbitrary sequence of probability measures $(\mu_n)_{n\geq d}$ on $\mathbf{R}^n$ (from the class we are considering), their projections to the coordinate subspace $\mathbf{R}^d$ must converge, as $n \to \infty$, to the standard Gaussian measure on $\mathbf{R}^d$.

A second step to prove Theorem 1 is based on the following statement ([B1], Corollary 4.3):

Lemma 3. *If a random variable ξ has distribution ν with log-concave density on the real line, then*

$$\frac{1}{12\,\mathrm{Var}(\xi)} \leq \lambda_1(\nu) \leq \frac{1}{\mathrm{Var}(\xi)}.$$

Together with (6) for $\xi(x) = |x|$, we thus get

$$\lambda_1(\nu) \geq \frac{1}{12}. \tag{9}$$

Proof of Theorem 1. We may assume that $n \geq 2$. As before, denote by ν the distribution of the Euclidean norm $\xi(x) = |x|$ under μ, and by σ_{n-1} the normalized Lebesgue measure on the unit sphere S^{n-1}. To prove the Poincaré-type inequality (1), take a smooth bounded function u on $\mathbf{R}^n$ and consider another smooth bounded function $v(r,\theta) = u(r\theta)$ on the product space $(0,+\infty) \times \mathbf{R}^n$. Under the product measure $\nu \times \sigma_{n-1}$, v has the same distribution as u has under μ.

By (9), the measure ν satisfies the Poincaré-type inequality on the line,

$$\mathrm{Var}_\nu(g) \leq 12 \int_0^{+\infty} |g'(r)|^2 d\nu(r),$$

where $g = g(r)$ is an arbitrary absolutely continuous function on $(0,+\infty)$. In particular, for $g(r) = v(r,\theta)$ with fixed $\theta \in S^{n-1}$, we get

$$\int_0^{+\infty} v(r,\theta)^2\, d\nu(r) \leq \left(\int_0^{+\infty} v(r,\theta)\, d\nu(r)\right)^2 + 12 \int_0^{+\infty} \left|\frac{\partial v}{\partial r}\right|^2 d\nu(r).$$

Now, $\frac{\partial v}{\partial r} = \langle \nabla u(r\theta), \theta\rangle$, so $\left|\frac{\partial v}{\partial r}\right| \leq |\nabla u(r\theta)|$. Integrating the above inequality over σ_{n-1}, we get

$$\int_{\mathbf{R}^n} u(x)^2\, d\mu(x) \leq \int_{S^{n-1}} w(\theta)^2\, d\sigma_{n-1}(\theta) + 12 \int_{\mathbf{R}^n} |\nabla u(x)|^2\, d\mu(x), \tag{10}$$

where $w(\theta) = \int_0^{+\infty} v(r,\theta)\, d\nu(r)$. For this function, which is well-defined and smooth on the whole space $\mathbf{R}^n$, the average over σ_{n-1} is exactly the average of u over μ. Hence, by the Poincaré inequality on the unit sphere,

$$\int_{S^{n-1}} w(\theta)^2\, d\sigma_{n-1}(\theta) \leq \left(\int_{\mathbf{R}^n} u(x)\, d\mu(x)\right)^2 + \frac{1}{n} \int_{S^{n-1}} |\nabla w(\theta)|^2 d\sigma_{n-1}(\theta). \tag{11}$$

(The classical Riemannian version of the Poincaré inequality is formulated for the "inner" gradient $\nabla_{S^{n-1}} w(\theta)$ on the unit sphere which is the projection of the usual gradient $\nabla w(\theta)$ onto the subspace orthogonal to θ. In this case the constant $\frac{1}{n}$ in (11) should be replaced with $\frac{1}{n-1}$.)

Since $\nabla w(\theta) = \int_0^{+\infty} r\, \nabla u(r\theta)\, d\nu(r)$, we have $|\nabla w(\theta)| \leq \int_0^{+\infty} r\, |\nabla u(r\theta)|\, d\nu(r)$. Hence, by the Cauchy–Bunyakovski inequality,

$$|\nabla w(\theta)|^2 \leq \int_0^{+\infty} r^2\, d\nu(r) \int_0^{+\infty} |\nabla u(r\theta)|^2\, d\nu(r) = n \int_0^{+\infty} |\nabla u(r\theta)|^2\, d\nu(r),$$

where we used the normalization condition $\mathbf{E}\xi^2 = n$. Together with (10) and (11), this estimate yields

$$\int_{\mathbf{R}^n} u(x)^2\, d\mu(x) \leq \left(\int_{\mathbf{R}^n} u(x)\, d\mu(x) \right)^2 + 13 \int_{\mathbf{R}^n} |\nabla u(x)|^2\, d\mu(x),$$

that is, the Poincaré-type inequality (1) with the lower bound $\lambda_1 \geq 1/13$.

The upper bound is trivial and follows by testing (1) on linear functions. This finishes the proof.

As already mentioned, the fact that (1) implies a concentration inequality, namely,

$$1 - \mu(A^h) \leq C e^{-c\sqrt{\lambda_1}\, h}, \quad h > 0, \quad \mu(A) \geq \frac{1}{2}, \tag{12}$$

where C and c are certain positive universal constants, was proved by M. Gromov and V.D. Milman, see [G-M]. They formulated it in the setting of a compact Riemannian manifold, but the assertion remains to hold in many other settings, e.g., for an arbitrary metric space (see e.g. [A-S], [B-L], [L]). The best possible constant in the exponent in (12) is $c = 2$ ([B2]), but this is not important for the present formulation of Theorem 1.

Remark. We do not know how to adapt the argument in order to prove, for all smooth u with μ-mean zero, a stronger inequality in comparison with (1),

$$c \int_{\mathbf{R}^n} |u(x)|\, d\mu(x) \leq \int_{\mathbf{R}^n} |\nabla u(x)|\, d\mu(x), \tag{13}$$

called sometimes a Cheeger-type inequality. On the shifted indicator functions $u = 1_A - \mu(A)$, (13) turns into an equivalent isoperimetric inequality for the μ-perimeter, $\mu^+(A) \geq 2c\, \mu(A)(1 - \mu(A))$. One deep conjecture ([K-L-S]) asserts that, for some universal $c > 0$, this isoperimetric inequality holds true under the isotropic condition (2) in the class of all log-concave measures μ. However, the hypothesis remains open even in the weaker forms such as Poincaré and concentration inequalities. And as we saw, already the particular case of a symmetrically log-concave measure leads to a rather sophisticated one-dimensional property such as Lemma 1.

Proof of Lemma 2. Let p be the probability density of η on $(0,+\infty)$. We apply the one-dimensional Prékopa–Leindler theorem (see [Pr1-2], [Le], or [Pi] for a short proof): given $t,s>0$ with $t+s=1$ and non-negative measurable functions u,v,w on $(0,+\infty)$ satisfying $w(tx+sy)\geq u^t(x)v^s(y)$, for all $x,y>0$, we have

$$\int_0^{+\infty} w(z)\,dz \geq \left(\int_0^{+\infty} u(x)\,dx\right)^t \left(\int_0^{+\infty} v(y)\,dy\right)^s. \tag{14}$$

Let $a>b>c>0$ and $b=ta+sc$. Since

$$\sup_{tx+sy=z} x^a y^c = a^{ta}c^{sc}\left(\frac{z}{ta+sc}\right)^{ta+sc},$$

the inequality (14) applies to $u(x)=(\frac{x}{a})^a p(x)$, $v(y)=(\frac{y}{c})^c p(y)$, and $w(z)=(\frac{z}{b})^b p(z)$. This is exactly what we need.

Remark. The multidimensional Prékopa–Leindler theorem yields a similar statement: For any random vector $(\eta_1,\dots,\eta_n)$ in $\mathbf{R}_+^n$ with log-concave distribution, the function $\varphi(a_1,\dots,a_n)=\mathbf{E}\,(\frac{\eta_1}{a_1})^{a_1}\dots(\frac{\eta_n}{a_n})^{a_n}$ is log-concave on $\mathbf{R}_+^n$.

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