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Large Deviations of Typical Linear Functionals on a Convex Body with Unconditional Basis

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Abstract. We study large deviations of linear functionals on an isotropic convex set with unconditional basis. It is shown that suitably normalized ℓ_1-balls play the role of extremal bodies.

1. Introduction

Let K be a convex body in $\mathbf{R}^n$ with the properties:

1) $\mathrm{vol}_n(K) = 1$, where vol_n stands for the Lebesgue measure;
2) given $x \in K$, $y \in \mathbf{R}^n$ such that $|y_j| \leq |x_j|$, for all $j \leq n$, we have $y \in K$;
3) the integrals

$$\int_K x_j^2 \, dx = L_K^2$$

do not depend on $j \leq n$.

By the assumption 2), the set K is centrally symmetric, and moreover, the canonical basis in $\mathbf{R}^n$ is unconditional for the norm associated to K. Under 2), the normalizing assumption 3) defines K as an isotropic body. This means, that linear functionals

$$f_\theta(x) = \theta_1 x_1 + \cdots + \theta_n x_n, \quad x \in \mathbf{R}^n,$$

parameterized by unit vectors $\theta = (\theta_1, \dots, \theta_n)$ have $L^2(K)$-norm equal to L_K.

Due to the hypotheses $1) - 3)$ on K, the quantity L_K satisfies $c_1 \leq L_K \leq c_2$, for some absolute constants $c_1, c_2 > 0$ (cf. [2]). Moreover, according to Borell's lemma ([4], Lemma 3.1), $L^p(K)$-norms of f_θ are at most Cp, for all $p \geq 1$ and some numerical constant C. This can be written in terms of the Young function $\psi_1(t) = e^{|t|} - 1$ and the corresponding Orlicz norm on K as one inequality

$$\|f_\theta\|_{\psi_1} \leq C_1.$$

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A natural general question in this direction (regarding of the unconditionality assumption) is how to determine whether or not, for some unit vector θ, or moreover, for most of them, we have a stronger inequality

$$\|f_\theta\|_{\psi_2} \le C_2 \tag{1.1}$$

with respect to the Young function $\psi_2(t) = e^{|t|^2} - 1$. The inequality (1.1) is equivalent to the property that f_θ admits a gaussian bound on the distribution of tails,

$$\mathrm{vol}_n\{x \in K : |f_\theta(x)| \ge t\} \le 2\, e^{-t^2/C}, \quad t \ge 0$$

(with C proportional to C_2). The study of this question was initiated by J. Bourgain [3] who related it to the slicing problem in Convex Geometry. While for this problem it is important to know how to control $\sup_\theta \|f_\theta\|_{\psi_2}$ as a quantity depending on K, it turns out non-trivial to see in general whether the inequality (1.1) holds true for at least one vector θ with a universal C_2 (a question posed and propagandized over the years by V. D. Milman). Recently, G. Paouris studied the problem for several families of isotropic bodies including zonoids and those that are contained in the Euclidean ball of radius of order $\sqrt{n}$. See [7] where one can also find further references and comments on the relationship to the slicing problem. In [1], it is shown that, under the hypotheses 1) – 3), the inequality (1.1) holds always true for the main direction, that is, for the functional

$$f(x) = \frac{x_1 + \dots + x_n}{\sqrt{n}}, \quad x \in \mathbf{R}^n.$$

In this paper we suggest another approach to this result which allows one to involve into consideration arbitrary linear functionals f_θ and thus to study their possible behavior on average.

Theorem 1.1. *For every vector $\theta \in \mathbf{R}^n$,*

$$\|f_\theta\|_{\psi_2} \le 4\, \|\theta\|_\infty \sqrt{3n}. \tag{1.2}$$

Here, $\|\theta\|_\infty = \max_{j \le n} |\theta_j|$. The inequality (1.2) may be applied to f itself which yields (1.1) with a dimension free constant.

Up to an absolute factor, the right hand in (1.2) cannot be improved. This can be shown on the example of the normalized ℓ^1-balls, see Proposition 2.1 below. On the other hand, the average value of $\|\theta\|_\infty \sqrt{n}$ with respect to the uniform measure σ_{n-1} on the unit sphere S^{n-1} is about $\sqrt{\log n}$. Therefore, one cannot hope that (1.1) will hold for most of the unit vectors in the sense of σ_{n-1}, so other norms or rates for distribution tails have to be examined in order to describe the (worst) typical behavior of linear functionals on K.

Theorem 1.2. *There exist positive numerical constants c_1, c_2 and t_0 with the following property. For all $\theta \in S^{n-1}$ except possibly for a set of σ_{n-1}-measure at most n^{-c_1},*

$$\mathrm{vol}_n\{x \in K : |f_\theta(x)| \ge t\} \le \exp\left\{-\frac{c_2\, t^2}{\log t}\right\}, \quad t \ge t_0. \tag{1.3}$$

Moreover, c_1 can be chosen arbitrarily large at the expense of suitable c_2 and t_0.

Thus, in the worst case, the tails of f_θ are"almost" Gaussian. In particular, for most unit vectors, we have a weakened version of (1.1),

$$\|f_\theta\|_{\psi_\alpha} \leq C_\alpha,$$

which is fulfilled for all $\alpha \in [1,2)$ with respect to the Young functions $\psi_\alpha(t) = e^{|t|^\alpha} - 1$ (with C_α depending on α, only).

Introduce the unit ball of the space ℓ_1^n,

$$B_1 = \{x \in \mathbf{R}^n : |x_1| + \dots + |x_n| \leq 1\}.$$

It is known that the basic assumptions $1) - 3)$ imply a set inclusion $K \subset CnB_1$, for some numerical C. This fact itself may inspire an idea that a number of essential properties of K could be inherited from the dilated ℓ_1^n-ball. One comparison claim of this kind is discussed in Section 3, where we also complete the proof of Theorems 1.1 and 1.2. The case of ℓ_1^n-ball has to be treated separately and is considered in Section 2.

2. Linear functionals on ℓ_1^n ball

Given a probability space (Ω, μ) and a Young function ψ on the real line $\mathbf{R}$ (i.e., a convex, even function such that $\psi(0) = 0$, $\psi(t) > 0$ for $t \neq 0$), one defines the corresponding Orlicz norm by

$$\|f\|_\psi = \|f\|_{L_\psi(\mu)} = \inf\left\{\lambda > 0 : \int \psi(f/\lambda)\, d\mu \leq 1\right\},$$

where f is an arbitrary measurable function on Ω. If $\psi(t) = |t|^p$ $(p \geq 1)$, we arrive at the usual Lebesgue space norm $\|f\|_p = \|f\|_{L^p(\mu)}$. It is well-known and easy to see that for $\psi = \psi_2$, the Orlicz norm $\|f\|_{\psi_2}$ is equivalent to $\sup_{p \geq 1} \frac{\|f\|_p}{\sqrt{p}}$. So, in order to get information on large deviations of f and, in particular, to bound its ψ_2-norm, it suffices to study the rate of growth of L^p-norms of f.

We equip $\Omega = B_1$, the unit ball of the space ℓ_1^n, with the uniform distribution μ_n. This probability measure has density

$$\frac{d\mu_n(x)}{dx} = \frac{n!}{2^n}\, 1_{B_1}(x), \quad x \in \mathbf{R}^n.$$

For any positive real numbers $p_1, \dots, p_n$, one has a well-known identity

$$\int_{\Delta_n} x_1^{p_1 - 1} \dots x_n^{p_n - 1}\, dx_1 \dots dx_n = \frac{\Gamma(p_1) \dots \Gamma(p_1)}{\Gamma(p_1 + \dots + p_n + 1)},$$

where the integration is performed over $\Delta_n = \{x \in \mathbf{R}_+^n : x_1 + \dots + x_n \leq 1\}$, the part of B_1 in the positive octant $\mathbf{R}_+^n = [0, +\infty)^n$. Together with the polynomial formula, this identity implies that, for any positive even integer $p = 2q$,

$$\int |f_\theta(x)|^{2q}\, d\mu_n(x) = \frac{n!\,(2q)!}{(n+2q)!} \sum_{q_1 + \dots + q_n = q} \theta_1^{2q_1} \dots \theta_n^{2q_n}, \tag{2.1}$$

where the summation is performed over all non-negative integers $q_1, \dots, q_n$ such that $q_1 + \dots + q_n = q$. One easily derives from this:

Proposition 2.1. *For every $\theta \in \mathbf{R}^n$,*

$$\frac{c_1 \, \|\theta\|_\infty}{\sqrt{n}} \leq \|f_\theta\|_{L_{\psi_2}(\mu_n)} \leq \frac{c_2 \, \|\theta\|_\infty}{\sqrt{n}}, \tag{2.2}$$

where c_1 and c_2 are absolute positive constants. One can take $c_1 = \frac{1}{\sqrt{6}}$, $c_2 = 2\sqrt{2}$.

In the sequel, we use notation $C_m^k = \frac{m!}{k!(m-k)!}$ for usual binomial coefficients (where $k = 0, 1, \dots m$).

Proof. From (2.1), setting $\alpha = \sqrt{n}\,\|\theta\|_\infty$, and recalling that the sum therein contains C_{n+q-1}^{n-1} terms, we get

$$\begin{aligned} \int |f_\theta|^{2q} \, d\mu_n \ &\leq \ \frac{n!\,(2q)!}{(n+2q)!} \, \frac{(n+q-1)!}{(n-1)!\,q!} \, \frac{\alpha^{2q}}{n^q} \\ &= \ \frac{n}{(n+2q)\dots(n+q)} \, \frac{C_{2q}^q \, q!\,\alpha^{2q}}{n^q} \ \leq \ \frac{4^q \, q!\,\alpha^{2q}}{n^{2q}}. \end{aligned} \tag{2.3}$$

Therefore, by Taylor's expansion, for all $|\lambda| < \frac{1}{2\alpha}$,

$$\begin{aligned} \int e^{(\lambda n f_\theta)^2} \, d\mu_n &= 1 + \sum_{q=1}^{\infty} \frac{\lambda^{2q}}{q!} \int |f_\theta|^{2q} \, d\mu_n \\ &\leq 1 + \sum_{q=1}^{\infty} 4^q \lambda^{2q} \alpha^{2q} = \frac{1}{1 - 4\lambda^2\alpha^2}. \end{aligned}$$

The last expression is equal to 2 for $\lambda = \frac{1}{2\sqrt{2}\alpha}$, so $n\|f_\theta\|_{L_{\psi_2}(\mu_n)} \leq 2\sqrt{2}\,\alpha$. This gives the upper estimate in (2.2) with $c_2 = 2\sqrt{2}$.

For the lower estimate, we may assume $\theta_j \geq 0$, for all $j \leq n$. It follows from (2.1) that all L^{2q}-norms $\theta \longrightarrow \|f_\theta\|_{2q}$ as functions of θ and therefore the function $\theta \longrightarrow \|f_\theta\|_{\psi_2}$ are coordinatewise increasing on $\mathbf{R}_+^n$. Consequently, $\|f_\theta\|_{\psi_2} \geq \|\theta\|_\infty \|f_1\|_{\psi_2}$ where $f_1(x) = x_1$. To bound from below $\|f_1\|_{\psi_2}$, note that, given a random variable ξ with $\|\xi\|_{\psi_2} = 1$, we always have $2 = \mathbf{E}\, e^{\xi^2} \geq \frac{1}{q!}\, \mathbf{E}\, \xi^{2q}$, so $\mathbf{E}\, \xi^{2q} \leq 2q!$ for all integers $q \geq 1$. Therefore, by homogeneity of the norm, $\mathbf{E}\, \xi^{2q} \leq 2q!\, \|\xi\|_{\psi_2}^{2q}$ in general. Applying this to $\xi = f_1$ and $q = n$, and recalling (2.1), we find that

$$\begin{aligned} \|f_1\|_{\psi_2}^{2n} &\geq \frac{1}{2n!} \int |f_1(x)|^{2n} \, d\mu_n(x) = \frac{1}{2n!} \, \frac{n!\,(2n)!}{(3n)!} \\ &= \frac{1}{2\,(2n+1)\dots(3n)} \geq \frac{1}{2\,(3n)^n}. \end{aligned}$$

Hence, $\|f_1\|_{\psi_2} \geq \frac{1}{\sqrt{6n}}$. This yields the lower bound in (2.2) with $c_1 = 1/\sqrt{6}$. $\square$

Now, we will try to sharpen the bound (2.3) on L^{2q}-norms. Given a vector $\theta \in S^{n-1}$, $n \geq 2$, put

$$C_n(\theta) = \|\theta\|_\infty \sqrt{\frac{n}{\log n}}.$$

Thus, $C_n(\theta) = \frac{1}{\sqrt{\log n}}$ for the main direction $\theta_j = \frac{1}{\sqrt{n}}$. However, this quantity is of order 1 for a typical θ, that is, when this vector is randomly selected with respect to the uniform distribution σ_{n-1} on the unit sphere S^{n-1}.

Proposition 2.2. *On (B_1, μ_n), for all $\theta \in S^{n-1}$ and every real $p \geq 1$,*

$$n \, \|f_\theta\|_p \leq C \max\left\{\sqrt{p},\, C_n(\theta)\sqrt{p \log p}\,\right\}, \tag{2.4}$$

where C is an absolute constant. One can take $C = 4.8$.

The inequality (2.4) implies the upper estimate of Proposition 2.1. Indeed, if $p \leq n$, we get

$$n \, \|f_\theta\|_p \leq C \max\left\{\sqrt{p},\, C_n(\theta)\sqrt{p \log n}\,\right\} = C \, \|\theta\|_\infty \sqrt{np},$$

where we also applied $1 \leq \|\theta\|_\infty \sqrt{n}$ (due to the normalization assumption $|\theta| = 1$). Thus,

$$\|f_\theta\|_p \leq \frac{C \, \|\theta\|_\infty}{\sqrt{n}} \sqrt{p}, \quad 1 \leq p \leq n.$$

The latter easily yields the right hand side of (2.2).

Introduce the full homogeneous polynomial of degree q,

$$P_q(a) \;=\; \sum_{q_1 + \dots + q_n = q} a_1^{q_1} \dots a_n^{q_n}, \quad a \in \mathbf{R}^n, \tag{2.5}$$

where the summation is performed over all non-negative integers $q_1, \dots, q_n$ such that $q_1 + \dots + q_n = q$. According to the exact formula (2.1), we need a more accurate estimate on $P_q(a)$ (with $a_i = \theta_i^2$) in comparison with what was used for the proof of Proposition 2.1 on the basis of the trivial bound $a_i \leq \|a\|_\infty$.

Lemma 2.3. *For any $q \geq 1$ integer and every $a \in \mathbf{R}^n$ such that $\sum_{i=1}^n |a_i| = 1$,*

$$P_q(a) \leq \left(2e \max\left\{\frac{1}{q},\, \|a\|_\infty\right\}\right)^q.$$

Proof. We may assume $a_i \geq 0$. First we drop the condition $\|a\|_1 = \sum_{i=1}^n |a_i| = 1$ but assume $a_i \leq \frac{1}{2}$, for all $i \leq n$. Then $\frac{1}{1-a_i} \leq e^{2a_i}$ and, performing the summation in (2.5) over all integers $q_i \geq 0$, we get

$$P_q(a) \leq \sum_{q_i \geq 0} a_1^{q_1} \dots a_n^{q_n} = \frac{1}{(1-a_1)\dots(1-a_n)} \leq e^{2\|a\|_1}.$$

Applying this estimate to the vector ta instead of a and assuming $\|a\|_1 = 1$, we thus obtain that

$$P_q(a) \le \frac{e^{2t}}{t^q}, \quad \text{whenever} \ \ 0 \le t \le \frac{1}{2\|a\|_\infty}.$$

Optimization over all admissible t leads to

$$P_q(a) \le \begin{cases} (\frac{2e}{q})^q, & \text{if} \ \ \|a\|_\infty \le \frac{1}{q}, \\ (2\|a\|_\infty)^q \exp\Big\{\frac{1}{\|a\|_\infty}\Big\}, & \text{if} \ \ \|a\|_\infty \ge \frac{1}{q}. \end{cases}$$

It remains to apply $\exp\{\frac{1}{\|a\|_\infty}\} \le e^q$ in the second case $\|a\|_\infty \ge \frac{1}{q}$. □

Proof of Proposition 2.2. Applying Lemma 2.3 and (2.1) with $a_i = \theta_i^2$, we obtain that, for every integer $q \ge 1$,

$$\|f_\theta\|_{2q}^{2q} \le \frac{n!\,(2q)!}{(n+2q)!} \left(2e \, \max\left\{\frac{1}{q}, \, \|a\|_\infty\right\}\right)^q.$$

Using $(n+2q)! \ge n!\,n^{2q}$ and $(2q)! \le (2q)^{2q}$, we thus get

$$\begin{aligned} n\,\|f_\theta\|_{2q} \ & \le \ 2q\sqrt{2e}\max\left\{\frac{1}{\sqrt{q}}, \, \sqrt{\|a\|_\infty}\right\} \\ & = \ 2\sqrt{2e}\max\{\sqrt{q}, \, q\,\|\theta\|_\infty\}. \end{aligned} \tag{2.6}$$

Now, starting from a real number $p \ge 2$, take the least integer q such that $p \le 2q$. Then, $q \le p$ and

$$\begin{aligned} n\,\|f_\theta\|_p \le n\,\|f_\theta\|_{2q} \ & \le \ 2\sqrt{2e}\max\{\sqrt{q}, \, q\,\|\theta\|_\infty\} \\ & \le \ 2\sqrt{2e}\max\{\sqrt{p}, \, p\,\|\theta\|_\infty\} \\ & = \ 2\sqrt{2e}\max\left\{\sqrt{p}, \, C_n(\theta)\,p\sqrt{\frac{\log n}{n}}\right\}. \end{aligned}$$

Assume $p \le n$. The function $\frac{\log x}{x}$ is decreasing in $x \ge e$, so $\frac{\log n}{n} \le \frac{\log p}{p}$ in case $p \ge e$. If $2 \le p \le e$, we have anyway $\frac{\log n}{n} \le \frac{\log 9}{\log 8}\,\frac{\log p}{p}$. Thus, in the range $2 \le p \le n$,

$$n\,\|f_\theta\|_p \le 2\sqrt{2e}\max\left\{\sqrt{p}, \, 1.03\,C_n(\theta)\,\sqrt{p\log p}\right\}.$$

For $p \ge n$, this inequality is immediate since then

$$n\,\|f_\theta\|_p \le n\,\|f_\theta\|_\infty = n\|\theta\|_\infty \le C_n(\theta)\,\sqrt{p\log p}.$$

It remains to note that $2\sqrt{2e}\,1.03 < 4.8$ and that the case $1 \le p \le 2$ follows from the estimate $n\,\|f_\theta\|_p \le n\,\|f_\theta\|_2 < \sqrt{2}$. Proposition 2.2 has been proved. □

Corollary 2.4. *For all $\theta \in S^{n-1}$ and every real $p \ge 2$,*

$$n\,\|f_\theta\|_p \ \le \ 7\max\{1, C_n(\theta)\}\sqrt{p\log p}. \tag{2.7}$$

3. The general case

Now, let us return to the general case of a body K in $\mathbf{R}^n$ satisfying the basic properties 1) – 3). As we already mentioned, within the class of such bodies, a suitably normalized ℓ_1^n-ball is the largest set. More precisely, we have

$$K \subset V \equiv Cn\, B_1,$$

for some universal constant C. As shown in [1], one may always take $C = \sqrt{6}$, and moreover $C = 1$, if additionally K is symmetric under permutations of coordinates. Note that the sets K and V have similar volume radii of order $\sqrt{n}$. Our nearest goal will be to sharpen this property in terms of distributions of certain increasing functionals with respect to the uniform distributions on these bodies which we denote by μ_K and μ_V, respectively.

Denote by $\mathcal{F}_n$ the family of all functions F on $\mathbf{R}^n$ that are symmetric about coordinate axes and representable on $\mathbf{R}^n_+$ as

$$F(x) = \pi\left([0,x_1] \times \cdots \times [0,x_n]\right), \quad x \in \mathbf{R}^n_+, \tag{3.1}$$

for some positive Borel measure π (finite on all compact subsets). If π is absolutely continuous, the second assumption on F is equivalent to the representation

$$F(x) = \int_0^{x_1} \dots \int_0^{x_n} q(t)\, dt, \quad x \in \mathbf{R}^n_+,$$

for some non-negative measurable function q. Of a special interest will be the functions of the form $F(x) = |x_1|^{p_1} \dots |x_n|^{p_n}$ with $p_1, \dots, p_n \geq 0$.

With these assumptions, we have:

Proposition 3.1. *If $F \in \mathcal{F}_n$, for all $t \geq 0$,*

$$\mu_K\{x \in \mathbf{R}^n \colon F(x) \geq t\} \leq \mu_V\{x \in \mathbf{R}^n \colon F(x) \geq t\}, \tag{3.2}$$

where the constant C defining V may be taken to be $\sqrt{6}$ (and one may take $C = 1$ in case K is symmetric under permutations of coordinates). In particular,

$$\int F(x)\, d\mu_K(x) \leq \int F(x)\, d\mu_V(x). \tag{3.3}$$

Proof. Consider

$$u(\alpha_1, \dots, \alpha_n) = \mu_K\{x \in K : |x_1| \geq \alpha_1, \dots, |x_n| \geq \alpha_n\}, \quad \alpha_j \geq 0.$$

By the Brunn–Minkowski inequality and since K is symmetric about coordinate axes, the function $u^{1/n}$ is concave on the convex set $K^+ = K \cap \mathbf{R}^n_+$. In addition, $u(0) = 1$. Without loss in generality, assume that u is continuously differentiable in K^+. Note that $\frac{\partial u(\alpha)}{\partial \alpha_j}\big|_{\alpha=0} = -\mathrm{vol}_{n-1}(K_j)$, where K_j are the sections of K by the coordinate hyperspaces $\{x \in \mathbf{R}^n : x_j = 0\}$. As known (cf. eg. [1]), for every hyperspace H in $\mathbf{R}^n$,

$$\mathrm{vol}_{n-1}(K \cap H) \geq c$$

with $c = \frac{1}{\sqrt{6}}$. Moreover, for all coordinate hyperspaces, one has a sharper estimate $\mathrm{vol}_{n-1}(K_j) \geq 1$, provided that K is invariant under permutations of coordinates. Consequently, $\frac{\partial u(\alpha)}{\partial \alpha_j}\big|_{\alpha=0} \leq -c$. By concavity, partial derivatives of $u^{1/n}$ are coordinatewise non-increasing, so, for all points α in K^+,

$$\frac{\partial u^{1/n}(\alpha)}{\partial \alpha_j} \leq \frac{\partial u^{1/n}(\alpha)}{\partial \alpha_j}\bigg|_{\alpha=0} = \frac{1}{n}\,\frac{\partial u(\alpha)}{\partial \alpha_j}\bigg|_{\alpha=0} \leq -\frac{c}{n}.$$

Thus, necessarily $u^{1/n}(\alpha) - u^{1/n}(0) \leq -\frac{c}{n}\,(\alpha_1 + \cdots + \alpha_n)$, that is,

$$\mu_K\{|x_1| \geq \alpha_1, \dots, |x_n| \geq \alpha_n\} \leq \left(1 - \frac{c\,(\alpha_1 + \cdots + \alpha_n)}{n}\right)^n. \tag{3.4}$$

Note that on the complement $\mathbf{R}^n_+ \setminus K^+$ the left hand side of (3.4) is zero, so this inequality extends automatically from K^+ to the whole octant $\mathbf{R}^n_+$.

Now, it is useful to observe that, whenever $\alpha_1 + \cdots + \alpha_n \leq Cn$, $\alpha_j \geq 0$,

$$\begin{aligned}
\mu_V\{|x_1| \geq \alpha_1, \dots, |x_n| \geq \alpha_n\} &= \frac{\mathrm{vol}_n\{x \in V : |x_1| \geq \alpha_1, \dots, |x_n| \geq \alpha_n\}}{\mathrm{vol}_n(V)} \\
&= \frac{\mathrm{vol}_n\left((Cn - \alpha_1 - \dots \alpha_n)\,B_1\right)}{\mathrm{vol}_n(CnB_1)} \\
&= \left(1 - \frac{\alpha_1 + \cdots + \alpha_n}{Cn}\right)^n.
\end{aligned}$$

Thus, (3.4) is exactly the desired inequality (3.2) with $C = \frac{1}{c}$ for the characteristic function $F_\alpha(x) = 1_{\{|x_1| \geq \alpha_1, \dots, |x_n| \geq \alpha_n\}}$. In the representation (3.1), this function corresponds to the measure π that assigns unit mass to the point α. To get (3.1) for all other $F's$, it remains to take into account an obvious fact that the functions F_α form a collection of all extremal "points" for the cone $\mathcal{F}_n$. Proposition 3.1 follows. □

As a basic example, we apply (3.3) to compare the absolute mixed moments of the measures μ_K and μ_V: for all $p_1, \dots, p_n \geq 0$,

$$\int |x_1|^{p_1} \dots |x_n|^{p_n}\, d\mu_K(x) \leq \int |x_1|^{p_1} \dots |x_n|^{p_n}\, d\mu_V(x). \tag{3.5}$$

As in case of the ℓ_1^n unit ball B_1, such moments are involved through the polynomial formula in the representation

$$\int |f_\theta|^{2q}\, d\mu = (2q)! \sum \frac{\theta_1^{2q_1} \dots \theta_n^{2q_n}}{(2q_1)! \dots (2q_1)!} \int x_1^{2q_1} \dots x_n^{2q_n}\, d\mu(x),$$

where q is a natural number, and where μ may be an arbitrary probability measure on $\mathbf{R}^n$ symmetric about the coordinate axes (as before, the summation performed over all non-negative integers $q_1, \dots, q_n$ such that $q_1 + \cdots + q_n = q$). In view of (3.5), we thus get:

Corollary 3.2. *For every $\theta \in \mathbf{R}^n$, and for all integers $q \geq 0$,*

$$\int |f_\theta|^{2q} \, d\mu_K \leq \int |f_\theta|^{2q} \, d\mu_V. \tag{3.6}$$

To complete the proof of Theorem 1.1, it remains to note that, with Taylor's expansion for $e^{(\lambda f_\theta)^2}$, (3.6) readily implies an analogous inequality for ψ_2-norm,

$$\|f_\theta\|_{L_{\psi_2}(\mu_K)} \leq \|f_\theta\|_{L_{\psi_2}(\mu_V)}.$$

Since $V = CnB_1$, the right hand side in terms of the uniform measure μ_n on B_1 is just $Cn \, \|f_\theta\|_{L_{\psi_2}(\mu_n)}$. Thus, by Proposition 2.1,

$$\|f_\theta\|_{L_{\psi_2}(\mu_K)} \leq 2\sqrt{2}\, C \, \|\theta\|_\infty \sqrt{n}.$$

The constant $2\sqrt{2}\, C$ does not exceed $4\sqrt{3}$, as it is claimed in (1.2), and does not exceed $2\sqrt{2}$ in case K is symmetric under permutations of the coordinates. This proves Theorem 1.1. □

Now, combining (3.6) with the moment estimate (2.6) on B_1, we obtain

$$\|f_\theta\|_{L^{2q}(\mu_K)} \leq 2C\sqrt{2e} \max\{\sqrt{q}, \, q \, \|\theta\|_\infty\}.$$

Moreover, with the same argument leading from (2.6) to (2.7) in the proof of Proposition of 2.2, the above estimate implies a precise anologue of Corollary 2.4, i.e., the inequality

$$\|f_\theta\|_{L^p(\mu_K)} \ \leq \ 7C \max\{1, C_n(\theta)\} \sqrt{p \log p}, \tag{3.7}$$

which holds true for every real $p \geq 2$ and for all $\theta \in S^{n-1}$ with $C_n(\theta) = \|\theta\|_\infty \sqrt{\frac{n}{\log n}}$. Also recall that we may take $C = \sqrt{6}$. To reach the statement of Theorem 1.2, one needs to transform (3.7) into an appropriate deviation inequality.

Lemma 3.3. *Given a measurable function ξ on some probability space $(\Omega, \mathbf{P})$, assume that its L^p-norms satisfy, for all $p \geq 2$ and some constant $A \geq 1/2$,*

$$\|\xi\|_p \leq A\sqrt{p \log p}. \tag{3.8}$$

Then, for all $t \geq 2Ae$,

$$\mathbf{P}\{|\xi| \geq t\} \leq \exp\left\{-\frac{t^2}{8A^2 e \, \log t}\right\}. \tag{3.9}$$

Proof. Put $\eta = \xi^2$ and write the assumption (3.8) as $\int \eta^q \, d\mathbf{P} \leq (Bq \log(2q))^q$, $q \geq 1$, where $B = 2A^2$. Hence, by Chebyshev's inequality, for any $x > 0$,

$$\mathbf{P}\{\eta \geq x\} \leq x^{-q} \int \eta^q \, d\mathbf{P} \leq \left(\frac{B\, q \log(2q)}{x}\right)^q.$$

Apply it to q of the form $\frac{cx}{\log(cx)}$, $c > 0$, to get

$$\mathbf{P}\{\eta \geq x\} \leq \left(Bc\, \frac{\log \frac{2cx}{\log(cx)}}{\log(cx)} \right)^{\frac{cx}{\log(cx)}}$$

Assume $cx \geq e$. Since the function $\frac{z}{\log z}$ increases in $z \geq e$, the requirement $q \geq 1$ is fulfilled and, in addition, $\log \frac{2cx}{\log(cx)} \leq \log(2cx) \leq 2\log(cx)$. Thus, we may simplify the above estimate as $\mathbf{P}\{\eta \geq x\} \leq (2Bc)^{\frac{cx}{\log(cx)}}$. Choosing $c = \frac{1}{2Be}$, we obtain that

$$\mathbf{P}\{\eta \geq x\} \leq \exp\left\{ -\frac{cx}{\log(cx)} \right\}, \quad \text{provided that } \ x \geq 2Be^2.$$

Equivalently, replacing x with t^2, $\mathbf{P}\{|\xi| \geq t\} \leq \exp\{-\frac{ct^2}{\log(ct^2)}\}$, for all $t \geq 2Ae$. Since $c = \frac{1}{4A^2 e} \leq 1$, we have $\log(ct^2) \leq \log(t^2)$, so

$$\mathbf{P}\{|\xi| \geq t\} \leq \exp\left\{ -\frac{ct^2}{2\log t} \right\}, \quad t \geq 2Ae,$$

which is the desired inequality (3.9). $\square$

Proof of Theorem 1.2. According to (3.7), for any $\theta \in S^{n-1}$, the linear functional $\xi = f_\theta$ on $(\Omega, \mathbf{P}) = (K, \mu_K)$ satisfies the assumption of Lemma 3.3 with constant $A(\theta) = 7\sqrt{6}\max\{1, C_n(\theta)\}$. As a function of θ, this constant has relatively small deviations with respect to the uniform measure σ_{n-1} on the sphere S^{n-1}. Indeed, consider the function $g(\theta) = \max_{j \leq n} |\theta_j|$. Since it has Lipschitz seminorm 1, by a concentration inequality on the sphere (cf. e.g. [6], [5]), for all $h > 0$,

$$\sigma_{n-1}\{g \geq m + h\} \leq e^{-nh^2/2}, \tag{3.10}$$

where m is σ_{n-1}-meadian for g. As is known, the median does not exceed $\alpha\sqrt{\frac{\log n}{n}}$, for some numerical $\alpha > 0$. Taking h proportional to m in (3.10), we obtain that $\sigma_{n-1}\{ g \geq \beta\sqrt{\frac{\log n}{n}} \} \leq n^{-(\beta-\alpha)^2/2}$, for every $\beta > \alpha$. Equivalently, in terms of $C_n(\theta)$, $\sigma_{n-1}\{C_n(\theta) \geq \beta\} \leq n^{-(\beta-\alpha)^2/2}$, so,

$$\sigma_{n-1}\{A(\theta) \geq 7\sqrt{6}\,\beta\} \leq \frac{1}{n^{(\beta-\alpha)^2/2}}.$$

Thus, starting with a constant $c_1 > 0$, take $\beta > \alpha$ such that $(\beta - \alpha)^2/2 = c_1$. Then, with $A = 7\sqrt{6}\,\beta$, we get, by Lemma 3.3,

$$\mu_K\{|f_\theta| \geq t\} \leq \exp\left\{ -\frac{t^2}{8A^2 e \, \log t} \right\}, \quad t \geq t_0 \equiv 2Ae.$$

This inequality holds true for all θ in S^{n-1} except for a set on the sphere of measure at most n^{-c_1}. $\square$

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