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LETTER TO THE EDITORS

COMPLEMENT TO THE PAPER "ON THE CENTRAL LIMIT THEOREM ALONG SUBSEQUENCES OF NONCORRELATED OBSERVATIONS"

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In this note we would like to add some additional historical remarks and references to our paper [5]. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with finite second moments, such that the following properties hold:

(a) $\mathbf{E}X_k X_j = \delta_{kj}$, for all k, j, where δ_{kj} denotes the Kronecker symbol;

(b) $(X_1^2 + \cdots + X_n^2)/n \Rightarrow \rho^2$ weakly in distribution, as $n \to \infty$, for some random $\rho \geqq 0$;

(c) $\max_{1 \leqq k \leqq n} |X_k| = o(\sqrt{n}\,)$, in probability, as $n \to \infty$.

Let $L^p = L^p(\Omega, \mathcal{F}, \mathbf{P})$ denote the space of p-integrable random variables with norm $\|\cdot\|_p$ $(1 \leqq p \leqq +\infty)$. Given a random variable $\rho \geqq 0$, defined possibly on a different probability space, we denote by $N(0, \rho^2)$ the distribution of ρZ, where Z is a standard normal random variable independent of ρ. At first we formulate an immediate corollary of Theorem 6.1 and Lemma 6.1 in [5] (appearing as Theorem 6.3 in the particular case $\rho = 1$).

THEOREM 1. *Assuming conditions* (a)–(c), *for an increasing sequence of indices* $\{i_n\}_{n \geqq 1}$, *we have weakly in distribution*

$$\frac{X_{i_1} + \cdots + X_{i_n}}{\sqrt{n}} \Rightarrow N(0, \rho^2). \tag{1}$$

Moreover, given a sequence $\{j_n\}_{n \geqq 1}$ *with* $\lim_{n\to\infty} j_n/n \to +\infty$, *there is a sequence* $\{i_n\}_{n \geqq 1}$ *satisfying*

$$i_n \leqq j_n \quad \textit{for all } n \textit{ large enough.} \tag{2}$$

The existence of a sequence satisfying (1), *without* the tightness property (2) for i_n, is known to hold in much more general situations and for several schemes of weighted sums. In particular, for sequences of real numbers $a = (a_n)$ such that $A_n = (a_1^2 + \cdots + a_n^2)^{1/2} \to +\infty$, $a_n = o(A_n)$ (which we will call an admissible sequence), one may consider the sequence of weighted sums $S_n(a) = (a_1 X_{i_1} + \cdots + a_n X_{i_n})/A_n$.

In 1955, G. W. Morgenthaler considered in [12] an arbitrary uniformly bounded orthonormal system X_n on the unit interval $\Omega = (0, 1)$ with the Lebesgue measure $\mathbf{P}$. He proved that there exist an increasing sequence i_n and a measurable function $\rho \geqq 0$ on Ω with $\|\rho\|_2 = 1$ and $\|\rho\|_\infty \leqq \sup_n \|X_n\|_\infty$ such that, for any admissible $a = (a_n)$,

$$S_n(a) \Rightarrow N(0, \rho^2). \tag{3}$$

Moreover, this convergence is stable in the sense that (3) holds on every measurable set B of positive measure with respect to the normalized restriction of $\mathbf{P}$ to B.

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More general statements, including necessary and sufficient conditions, were obtained in the mid 1960s by V. F. Gaposhkin in a series of papers [7], [8], and [9], and later in [10]. Here we formulate two theorems.

THEOREM 2 (see [7, Theorem 5], [10, Theorem 6]). *Let* $\mathbf{E}X_n^2 = 1$, $n \geqq 1$, *and let* $\rho \geqq 0$ *be a random variable with* $\|\rho\|_2 = 1$. *The following properties are equivalent*:

1) *There exists an increasing sequence* i_n *such that the central limit theorem* (3) *holds for any admissible* $a = (a_n)$;

2) *there exists an increasing sequence* i_n *such that* $X_{i_n} \to 0$ *weakly in* L^2 *and* $X_{i_n}^2 \to \rho_0^2$ *weakly in* L^1 *as* $n \to \infty$, *for some* $\rho_0 \geqq 0$ *on* Ω *equidistributed with* ρ.

Here the case where $\rho = 1$ when the limit distribution is standard normal is of special interest. The existence of a random variable ρ with $\|\rho\|_2 = 1$ such that (3) holds for some increasing sequence i_n is equivalent to the weak convergence $X_{i_n} \to 0$ together with the uniform integrability of the sequence $X_{i_n}^2$ (cf. [7, Theorem 8] or [8, Theorem 1.5.3]).

THEOREM 3 (see [8, Theorem 1.5.2], [10, Theorem 5]). *If* $X_n \to 0$ *weakly in* L^2, *then there exists an increasing sequence* i_n *such that the central limit theorem* (3) *holds for any admissible* $a = (a_n)$ *and some random variable* $\rho \geqq 0$.

In [10] Gaposhkin introduced an "equivalence lemma," which allowed us to reduce many problems on subsequences of X_n to martingale differences (such as convergence of series, the central limit theorem, the law of the iterated logarithm) and eventually to extend the corresponding statements from Lebesgue measure and the space $\Omega = (0, 1)$ to arbitrary probability spaces.

Theorem 3 was rediscovered by S. D. Chatterji [6], with a similar martingale approach. Chatterji introduced an informal statement known as principle of subsequences; it states that any limit theorem about independent, identically distributed random variables continues to hold under proper moment assumptions for a certain subsequence of a given sequence of random variables. This general observation was made precise and developed by D. J. Aldous [1], and later by I. Berkes and E. Péter [2].

One should note, however, that not much is known about the speed of increase of the subsequence chosen to satisfy a central limit theorem. For example, sharpening a classical result of R. Salem and A. Zygmund [13] on lacunary trigonometric subsequences, P. Erdös [11] proved that any sequence $X_{i_n}(\omega) = \cos(2\pi i_n \omega)$ on $\Omega = (0, 1)$ with $i_{n+1}/i_n \geqq 1 + c_n/\sqrt{n}$, $c_n \to +\infty$, satisfies (1) with the standard normal limit distribution; cf. also [3]. Note that here i_n must grow faster than $e^{\sqrt{n}}$. Using randomization of indices in a trigonometric orthonormal system, Berkes [4] showed that a sequence i_n in (1) with $\rho = 1$ can be chosen to satisfy $i_{n+1} - i_n = O(j_n)$, for any prescribed $j_n \to +\infty$. Theorem 1 above describes a similar property, with possible application to other orthonormal systems.

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