On Isoperimetric Constants for Log-Concave Probability Distributions*

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Summary. Lower bounds on the isoperimetric constant for logarithmically concave probability measures are considered in terms of the distribution of the Euclidean norm. A refined form of Kannan–Lovász–Simonovits' inequality is obtained.

Given a Borel probability measure μ on $\mathbf{R}^n$, its isoperimetric constant or, isoperimetric coefficient, is defined as the optimal value $h = h(\mu)$ satisfying an isoperimetric-type inequality

$$\mu^+(A) \geq h \, \min\big\{\mu(A), 1 - \mu(A)\big\}. \tag{1}$$

Here, A is an arbitrary Borel subset of $\mathbf{R}^n$ of measure $\mu(A)$ with μ-perimeter $\mu^+(A) = \lim_{\varepsilon \downarrow 0} \frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon}$, where $A_\varepsilon = \{x \in \mathbf{R}^n : |x - a| < \varepsilon, \text{ for some } a \in A\}$ denotes an open ε-neighbourhood of A with respect to the Euclidean distance.

The quantity $h(\mu)$ represents an important geometric characteristic of the measure and is deeply related to a number of interesting analytic inequalities. As an example, one may consider a Poincaré-type inequality

$$\int |\nabla f|^2 \, d\mu \;\geq\; \lambda_1 \int |f|^2 \, d\mu$$

in the class of all smooth functions f on $\mathbf{R}^n$ such that $\int f \, d\mu = 0$. The optimal value λ_1, the so-called spectral gap, satisfies $\lambda_1 \geq h^2/4$. This relation goes back to the work by J. Cheeger in the framework of Riemannian manifolds [C] and – in a more general form – to earlier works by V.G. Maz'ya (cf. [M1-2], [G]). The problem on bounding these two quantities from below has a long story. In this note we specialize to the class of log-concave probability measures, in which case, as was recently shown by M. Ledoux [L], λ_1 and h are equivalent ($\lambda_1 \leq 36\, h^2$).

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Following A. Prékopa [P], μ is called logarithmically concave (or, log-concave), if for all non-empty convex sets A, B in $\mathbf{R}^n$, and $t \in (0,1)$,

$$\mu\big((1-t)A + tB\big) \geq \mu(A)^{1-t}\mu(B)^t,$$

where $(1-t)A + tB = \{(1-t)a + tb : a \in A,\, b \in B\}$ denotes the Minkowski average. The definition reduces to the statement that μ is concentrated on some affine subspace L of $\mathbf{R}^n$, where it is absolutely continuous with respect to Lebesgue measure and has a density p, satisfying

$$p\big((1-t)x + ty\big) \geq p(x)^{1-t}p(y)^t, \quad \text{for all } \ x, y \in L,\ t \in (0,1). \qquad (2)$$

For example, a uniform distribution over an arbitrary convex body K in $\mathbf{R}^n$ is log-concave (the convex body case). For a full description, including more general classes of convex measures, see C. Borell [Bor1-2]. In the sequel, by saying that μ is k-dimensional, we mean that the supporting subspace L has dimension k.

For any log-concave probability measure μ, its isoperimetric constant is positive and may be bounded from below, up to some universal constant $c > 0$, as

$$h(\mu) \geq \frac{c}{\int |x|\, d\mu(x)}. \qquad (3)$$

This inequality was obtained by R. Kannan, L. Lovász and M. Simonovits for the convex body case as part of the study of randomized volume algorithms ([K-L-S], Main Theorem). Actually, their proof based on a localization lemma of Kannan and Lovász [K-L] may easily be extended to the general log-concave case. A different approach, using the Prékopa–Leindler functional form for the Brunn–Minkowski inequality, was later proposed in [B1].

Our aim is to get the following sharpening of the bound (3) involving the distribution of the Euclidean norm. Let $X = (X_1, \dots, X_n)$ be a random vector in $\mathbf{R}^n$ with distribution μ, and $|X| = (X_1^2 + \dots + X_n^2)^{1/2}$ be its Euclidean length.

Theorem 1. *If μ is log-concave, then*

$$h(\mu) \geq \frac{c}{\mathrm{Var}(|X|^2)^{1/4}}, \qquad (4)$$

where c is a universal constant.

Here, $\mathrm{Var}(|X|^2) = \mathbf{E}\,|X|^4 - (\mathbf{E}\,|X|^2)^2 = \int |x|^4\, d\mu(x) - (\int |x|^2\, d\mu(x))^2$ is the variance of $|X|^2$.

By Borell's lemma ([Bor1], Lemma 3.1), L^p-norms of $|X|$ are equivalent, so $\mathbf{E}|X|^4 \leq C^4\, (\mathbf{E}\,|X|)^4$, for some positive numerical constant C. Therefore,

$$\mathrm{Var}\big(|X|^2\big)^{1/4} \leq \big(\mathbf{E}|X|^4\big)^{1/4} \leq C\,\mathbf{E}\,|X|,$$

and thus (4) implies the K-L-S bound (3). To see that there can be an essential difference between (3) and (4), take the unit ball B in $\mathbf{R}^n$ with center at the origin and equip it with the normalized Lebesgue measure μ. Then, $|X|$ has the distribution function $F_n(t) = \mathbf{P}\{|X| \leq t\} = t^n$, $0 \leq t \leq 1$, so $\mathbf{E}\,|X| = \int_0^1 t\,dF_n(t) = \frac{n}{n+1}$. Hence, the right hand side of (3) is of order 1. On the other hand,

$$\mathrm{Var}(|X|^2) = \int_0^1 t^4\,dF_n(t) - \left(\int_0^1 t^2\,dF_n(t)\right)^2 = \frac{4n}{(n+2)^2(n+4)},$$

so, the right hand side of (4) is of order $\sqrt{n}$. Hence, in this case (4) provides a correct estimate for $h(\mu)$ with respect to the dimension n. Equivalently, if B is a ball of volume radius of order 1, then $h(\mu)$ is of order 1, as well.

More generally, suppose the measure μ is normalized to be in isotropy position in the sense that

$$\mathbf{E}\,\langle X,\theta\rangle^2 = \int \langle x,\theta\rangle^2\,d\mu(x) = |\theta|^2, \qquad \theta \in \mathbf{R}^n. \tag{5}$$

Then, $\mathbf{E}\,|X| \leq (\mathbf{E}|X|^2)^{1/2} = \sqrt{n}$, and (3) leads to $h(\mu) \geq c/\sqrt{n}$. It is unknown whether this bound can be improved in general. Nevertheless, by virtue of Theorem 1, one may reach an improvement for some classes of measures (or bodies). For example, one interesting class is described by the condition

$$\mathbf{E}\,X_i^2 X_j^2 \leq \mathbf{E}\,X_i^2\,\mathbf{E}\,X_j^2, \qquad i \neq j. \tag{6}$$

That is, a log-concave probability measure μ on $\mathbf{R}^n$ belongs to this class, if the squares of the coordinates have non-positive covariances $\mathrm{cov}(X_i^2, X_j^2)$. In this case, if $\mathbf{E}X_i^2 = 1$ for all $i \leq n$ (which holds under the isotropy assumption), we have that

$$\mathrm{Var}(|X|^2) = \sum_{i=1}^n \mathrm{Var}(X_i^2) + 2\sum_{i<j}\mathrm{cov}(X_i^2, X_j^2) \leq \sum_{i=1}^n \mathbf{E}\,X_i^4 \leq Cn,$$

for some universal constant C. Therefore, Theorem 1 yields:

Corollary 1. *If a log-concave isotropic measure μ on $\mathbf{R}^n$ satisfies* (6)*, then*

$$h(\mu) \geq \frac{c}{n^{1/4}}, \tag{7}$$

where c is a universal constant.

As a more specific case, consider the uniform distribution μ on the dilated ℓ_p^n-ball

$$K = \left\{x \in \mathbf{R}^n : |x_1|^p + \cdots + |x_n|^p \leq c^p\right\}$$

with parameter $1 \leq p \leq +\infty$ and with $c = c(p,n)$ chosen to satisfy the isotropy condition (5) (c is of order $n^{1/p}$). That the covariance property (6) is fulfilled for such a family of convex bodies was observed by K. Ball and I. Perissinaki [B-P]. As we mentioned, in the case $p = 2$, $h(\mu)$ is of order 1. The same is true for $p = +\infty$ (H. Hadwiger) and for the whole range $2 \le p \le +\infty$, since then μ can be obtained from the canonical Gaussian measure as Lipschitz transform. When $1 \le p < 2$, the correct asymptotic with respect to the dimension seems to be unknown, and we can only state that $h(\mu) \ge c\,n^{-1/4}$. In this case, the constant is also believed to be of order 1; at least, this is inspired by concentration results, obtained by G. Schechtman and J. Zinn [S-Z]. More generally, Kannan, Lovász and Simonovits conjectured that $h(\mu)$ is of order 1 for arbitrary isotropic convex bodies.

Now, let us turn to the proof of Theorem 1. We use the localization argument of [K-L-S], but choose a somewhat different hypothesis in applying the localization lemma. The argument goes back to the bisection method of L. E. Payne and H. F. Weinberger [P-W]; similar ideas were also developed by M. Gromov and V. D. Milman in [G-M]; cf. also [A] and [F-G1,2]. Below we state as a lemma a slightly modified variant of Corollary 2.2 appearing in [K-L-S].

Lemma 1. *Let $\alpha, \beta > 0$, and suppose u_i, $i = 1,2,3,4$, are non-negative continuous functions on $\mathbf{R}^n$ such that for any segment $\Delta \subset \mathbf{R}^n$ and any affine function ℓ on Δ,*

$$\left(\int_\Delta u_1\, e^\ell\right)^\alpha \left(\int_\Delta u_2\, e^\ell\right)^\beta \le \left(\int_\Delta u_3\, e^\ell\right)^\alpha \left(\int_\Delta u_4\, e^\ell\right)^\beta. \tag{8}$$

Then,

$$\left(\int_{\mathbf{R}^n} u_1\right)^\alpha \left(\int_{\mathbf{R}^n} u_2\right)^\beta \le \left(\int_{\mathbf{R}^n} u_3\right)^\alpha \left(\int_{\mathbf{R}^n} u_4\right)^\beta. \tag{9}$$

The one-dimensional integrals in (8) are taken with respect to Lebesgue measure on Δ, while the integrals in (9) are n-dimensional.

It should be clear that Lemma 1 remains to hold for many discontinuous functions u_i, as well, like the indicator functions of open or closed sets in the space. For the uniform distribution μ on a convex body K in $\mathbf{R}^n$, the approach of [K-L-S] is to apply the lemma with $\alpha = \beta = 1$ to the functions of the form

$$u_1 = 1_A, \quad u_2 = 1_B, \quad u_3 = 1_C, \quad u_4(x) = \frac{\mathrm{const}\,|x|}{\varepsilon}\, 1_K(x),$$

where A and B are arbitrary "regular" disjoint subsets of $\mathbf{R}^n$ at the distance $\varepsilon = \mathrm{dist}(A,B)$ and where $C = \mathbf{R}^n \setminus (A \cup B)$. Then (9) turns into

$$\mu(A)\mu(B) \le \mu(C)\, \frac{\mathrm{const}}{\varepsilon} \int |x|\, d\mu(x), \tag{10}$$

and letting $\varepsilon \to 0$, we arrive at the desired isoperimetric inequality (1) with $\frac{1}{h} = 2\,\mathrm{const} \int |x|\, d\mu(x)$. On the other hand, (8) turns into a one-dimensional inequality which is similar to (10). The only difference is that μ should be replaced by a specific probability measure μ_ℓ concentrated on Δ and having, up to a normalizing constant, the density e^ℓ with respect to Lebesgue measure on Δ. That is how, the bound (3) reduces to the one-dimensional inequality (10) in the body case.

More generally, if μ is absolutely continuous and has a density p satisfying (2), then in (8) we are dealing with a probability measure μ_ℓ, concentrated on Δ and having, up to a normalizing constant, the density pe^ℓ. It satisfies (2), so the bound (3), being stated for the class of all absolutely continuous log-concave probability measures on $\mathbf{R}^n$, may also be reduced to the inequality (10) about arbitrary log-concave measures on Δ. Therefore, we obtain the following corollary from Lemma 1:

Corollary 2. *Let g be a non-negative continuous function on $\mathbf{R}^n$. Let A, B be open disjoint subsets of $\mathbf{R}^n$ at distance $\varepsilon = \mathrm{dist}(A,B)$, and put $C = \mathbf{R}^n \setminus (A \cup B)$. If the inequality*

$$\mu(A)\mu(B) \leq \frac{\mu(C)}{\varepsilon} \int g \, d\mu \tag{11}$$

holds for any one-dimensional log-concave probability measure, then it holds for any n-dimensional log-concave probability measure on $\mathbf{R}^n$.

In the conclusion, the dimension is irrelevant and can be ignored.

Also, as we already mentioned, letting $\varepsilon \to 0$, (11) takes the form of an isoperimetric inequality

$$\mu(A)\mu(B) \leq \mu^+(C) \int g \, d\mu. \tag{12}$$

Actually, it is easy to show that (12) is equivalent to (11) when these inequalities are required to hold for all admissible partitions A, B, C (see e.g. [B-Z], Proposition 10.1). Recalling the definition (1) and using $2\mu(A)\mu(B) \geq \max\{\mu(A), \mu(B)\}$, one may reformulate Corollary 2 equivalently up to a factor as:

Corollary 3. *Given a non-negative continuous function g on $\mathbf{R}^n$, if the inequality $\frac{1}{h(\mu)} \leq \int g \, d\mu$ is fulfilled for any one-dimensional log-concave probability measure μ, then for any log-concave probability measure μ on $\mathbf{R}^n$, we have $\frac{1}{h(\mu)} \leq 2 \int g \, d\mu$.*

Proof of Theorem 1. If ξ is a random variable with a log-concave distribution μ on the real line, then

$$c_1 \sqrt{\mathrm{Var}(\xi)} \leq \frac{1}{h(\mu)} \leq c_2 \sqrt{\mathrm{Var}(\xi)}. \tag{13}$$

The optimal constants, which are not important for us, are $c_1 = 1/\sqrt{2}$, $c_2 = \sqrt{3}$ (cf. [B1], Proposition 4.1). Any one-dimensional log-concave probability measure μ on $\mathbf{R}^n$ may be viewed as the distribution of a random vector $a+\xi\theta$, where a, θ are orthogonal vectors, $|\theta| = 1$, and ξ is a random variable with a log-concave distribution. Clearly, μ also satisfies (13). Hence, by Corollary 3, if the inequality

$$\sqrt{\mathrm{Var}(\xi)} \ \le \ \mathbf{E} g(a+\xi\theta) \tag{14}$$

holds for all ξ as above and for all vectors a, θ in $\mathbf{R}^n$, such that $\langle a, \theta\rangle = 0$, then

$$\frac{1}{h(\mu)} \ \le \ 2\, c_2 \int g\, d\mu \tag{15}$$

in the class of all log-concave probability measures μ on $\mathbf{R}^n$. We choose $g(x) = C\,\big|\,|x|^2 - \alpha\big|^{1/2}$ with an arbitrary number α, but with a constant C to be specified. In this case, the quantity $\mathbf{E} g(a+\xi\theta) = C\mathbf{E}\big|\,|a|^2+\xi^2-\alpha\big|^{1/2}$ satisfies (14) in view of the equivalence of L^p-norms of polynomials with respect to log-concave distributions. To be more precise, if Q is a polynomial on $\mathbf{R}^n$ of degree d, and μ is a log-concave probability measure, then for $\|Q\|_p = (\int |Q|^p\, d\mu)^{1/p}$ there is the relation

$$\|Q\|_p \ \le \ c(d,p)\, \|Q\|_0, \quad p \ge 0, \tag{16}$$

with constants $c(d,p)$ depending on d and p, only (cf. [Bou], [B2], [B-G]). In particular, $\|Q\|_2 \le c\, \|Q\|_{1/2}$ for any quadratic function Q with $c = c(2,2)$. Therefore,

$$c\big(\mathbf{E}\,\big|\,|a|^2+\xi^2-\alpha\big|^{1/2}\big)^2 \ge \big(\mathbf{E}\,\big|\,|a|^2+\xi^2-\alpha\big|^2\big)^{1/2} \ge \mathrm{Var}(\xi^2)^{1/2}. \tag{17}$$

Also, if $L^2 = \mathbf{E}\xi^2$, we have

$$\begin{aligned} \mathrm{Var}(\xi^2)^{1/2} &\ge \|\xi^2 - L^2\|_0 \ = \ \|\xi - L\|_0\, \|\xi + L\|_0 \\ &\ge \frac{1}{c^2}\|\xi - L\|_2\, \|\xi + L\|_2 \ \ge \ \frac{1}{c^2}\mathrm{Var}(\xi), \end{aligned}$$

where we applied (16) once more on the last step. Together with (17) this yields

$$C\,\mathbf{E}\,\big|\,|a|^2+\xi^2-\alpha\big|^{1/2} \ge \sqrt{\mathrm{Var}(\xi)}$$

with $C = c^{3/2}$, so the hypothesis (14) is fulfilled.

Now, let's look at the conclusion (15). If X is a random vector with distribution μ, by Jensen's inequality,

$$\int g\, d\mu = C\,\mathbf{E}\,\big|\,|X|^2 - \alpha\big|^{1/2} \le C\ \big(\mathbf{E}\,\big|\,|X|^2-\alpha\big|^2\big)^{1/4}.$$

The right hand side is minimized for $\alpha = \mathbf{E}|X|^2$ and becomes $C\,\mathrm{Var}(|X|^2)^{1/4}$. Hence, $\frac{1}{h(\mu)} \le 2\,c_2 C\,\mathrm{Var}(|X|^2)^{1/4}$ which is the claim.

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