A Remark on the Surface Brunn–Minkowski-Type Inequality*

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In this note we would like to attract reader's attention to the following functional form for the surface Brunn–Minkowski-type inequality.

Theorem 1. *Let $0 < t < 1$ and let u, v, w be non-negative, quasi-concave, smooth functions on $\mathbf{R}^n$, such that $w(x) \to 0$, as $|x| \to \infty$, and*

$$w\big(tx + (1-t)y\big) \geq u(x)^t v(y)^{1-t}, \tag{1}$$

for all $x, y \in \mathbf{R}^n$. Then

$$\int |\nabla w(z)|\, dz \geq \left(\int |\nabla u(x)|\, dx\right)^t \left(\int |\nabla v(y)|\, dy\right)^{1-t}. \tag{2}$$

A function w is called quasi-concave, if $w(tx+(1-t)y) \geq \min\{w(x), w(y)\}$, whenever $x, y \in \mathbf{R}^n$ and $0 < t < 1$ (cf. e.g. [C-F] for an account on equivalent definitions and basic properties of such functions.) In particular, all log-concave functions are quasi-concave. In this case, the assumption on smoothness may be removed from the hypotheses of Theorem 1.

Let A and B be convex bodies in $\mathbf{R}^n$. Approximating these sets by smooth log-concave functions u and v, inequality (2) yields

$$S\big(tA + (1-t)B\big) \ \geq\ S(A)^t S(B)^{1-t}, \tag{3}$$

and by homogeneity, for $n \geq 2$,

$$S\big(tA + (1-t)B\big) \ \geq\ \Big[t\, S(A)^{1/(n-1)} + (1-t)\, S(B)^{1/(n-1)}\Big]^{n-1}, \tag{4}$$

where we use $S(\cdot)$ to denote the area size of the surface of a corresponding convex body. This is a Brunn–Minkowski-type inequality for the functional S, cf. [S]. The bound (4) is optimal in the sense that its right hand side

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provides minimum of $S(tA + (1-t)B)$ in terms of $S(A)$ and $S(B)$; however, the advantage of the "log-concave" form (3) is that it remains to formally hold when one of the sets is empty.

Thus, inequality (2) under the hypothesis (1) may be viewed as a functional form for (4). The class of quasi-concave functions is natural in Theorem 1, since only for such functions the sets of the form $A_u(\lambda) = \{x \in \mathbf{R}^n : u(x) \geq \lambda\}$ are convex, and since (4) is stated for convex sets.

Recall that, for all non-negative measurable functions u, v, and w, satisfying the condition (1), we have the Prékopa–Leindler inequality ([Pr1-2], [L])

$$\int w(z)\,dz \geq \left(\int u(x)\,dx\right)^t \left(\int v(y)\,dy\right)^{1-t}, \tag{5}$$

which represents a natural functional form for the volume Brunn–Minkowski inequality

$$\mathrm{vol}_n\big(tA + (1-t)B\big) \;\geq\; \Big[t\,\mathrm{vol}_n(A)^{1/n} + (1-t)\,\mathrm{vol}_n(B)^{1/n}\Big]^n. \tag{6}$$

Other functional forms of (6), earlier references and discussion of history may be found in S. Das Gupta [DG] and R. Gardner [G]. Prékopa–Leindler's theorem has found a number of interesting applications in Convex Geometry and Analysis; let us only mention the works by C. Borell [Bo1-2], K. Ball [Ba] and B. Maurey [M]. In fact, (5) being combined with (3) may also be used to derive inequality (2).

Indeed, assume the functions u and v are not identically zero, so that both vanish as $|x| \to \infty$, since w does. Hence, the sets $A_u(\lambda) = \{u \geq \lambda\}$, $\lambda > 0$, are convex, bounded (and perhaps empty), and similarly for v and w.

Now, by the hypothesis (1),

$$tA_u(\lambda_1) + (1-t)A_v(\lambda_2) \subset A_w(\lambda_1^t \lambda_2^{1-t}), \qquad \lambda_1, \lambda_2 > 0,\ t \in (0,1),$$

as long as both $A_u(\lambda_1)$ and $A_v(\lambda_2)$ are non-empty. Anyhow, by (3) and by monotonicity of S, the functions

$$f(\lambda) = S\big(A_u(\lambda)\big), \quad g(\lambda) = S\big(A_v(\lambda)\big), \quad h(\lambda) = S\big(A_w(\lambda)\big)$$

satisfy $h(\lambda_1^t \lambda_2^{1-t}) \geq f(\lambda_1)^t g(\lambda_2)^{1-t}$, for all $\lambda_1, \lambda_2 > 0$. This property is a multiplicative version of (1) in dimension one, and it also implies (5), ([Ba], Lemma 3), i.e.,

$$\int_0^{+\infty} h(\lambda)\,d\lambda \geq \left(\int_0^{+\infty} f(\lambda)\,d\lambda\right)^t \left(\int_0^{+\infty} g(\lambda)\,d\lambda\right)^{1-t}.$$

Finally, applying the coarea formula $\int_{\mathbf{R}^n} |\nabla u(x)|\,dx = \int_0^{+\infty} f(\lambda)\,d\lambda$ to u, as well as to the functions v and w, we arrive at the desired conclusion (2).

More generally, with a similar argument one may consider Choquet's integrals $\int \varphi\,d\mu \equiv \int_0^{+\infty} \mu\{\varphi \geq \lambda\}\,d\lambda$ for $\varphi \geq 0$ with an arbitrary monotone set function $\mu \geq 0$ on $\mathbf{R}^n$, such that $\mu(tA + (1-t)B) \geq \mu(A)^t \mu(B)^{1-t}$ in the class of all convex bodies in the n-space. Under the assumptions of Theorem 1, one then gets that

$$\int w \, d\mu \geq \left(\int u \, d\mu \right)^t \left(\int v \, d\mu \right)^{1-t}. \tag{7}$$

For example, the p-capacity $\mu(A) = \inf\{\int |\nabla g(x)|^p \, dx : g \geq 1_A,\ g \in C_0^\infty(\mathbf{R}^n)\}$ is included in (7) whenever $1 \leq p < n$. In that case, the log-concavity of μ was proved by C. Borell [Bo2] for $p = 2$, $n \geq 3$ (the case of Newton capacity) and by A. Colesanti and P. Salani for all $p < n$. When $p = 1$, (7) coincides with (2). On the other hand, when μ is Lebesgue measure, we return to (5).

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