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On weighted isoperimetric and Poincaré-type inequalities

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Abstract: Weighted isoperimetric and Poincaré-type inequalities are studied for κ-concave probability measures (in the hierarchy of convex measures).

1. Introduction

A Borel probability measure μ on $\mathbb{R}^n$ is said to satisfy a weighted Poincaré-type inequality with weight function w^2 (where w is a fixed non-negative, Borel measurable function), if for any bounded smooth function f on $\mathbb{R}^n$ with gradient ∇f,

$$\mathrm{Var}_\mu(f) \leq \int |\nabla f|^2 w^2 d\mu. \tag{1.1}$$

As usual, $\mathrm{Var}_\mu(f) = \int f^2 d\mu - (\int f d\mu)^2$ stands for the variance of f under μ.

As a classical example, the standard Gaussian measure $\mu = \gamma_n$ with density $\frac{d\gamma_n(x)}{dx} = (2\pi)^{-n/2}\, e^{-|x|^2/2}$ with respect to Lebesgue measure on $\mathbb{R}^n$ satisfies (1.1) with $w = 1$. In general, the validity of (1.1) with a constant weight, that is, the usual Poincaré-type inequality requires that all Lipschitz functions have finite exponential moments under the underlying measure μ (cf. [1, 24, 29, 30]). So, in order to involve in (1.1) other important distributions, it is natural to allow non-constant weight functions. This way one may analyze concentration properties and the behaviour of the associated Markov semigroups for various probability measures that have rather heavy (e.g. polynomial) tails at infinity. Another closely related family of analytic inequalities that serve the same aim are the so-called weak Poincaré and logarithmic Sobolev inequalities with an oscillation term, intensively studied in the recent years (cf. [3, 4, 9, 17, 39]). We do not touch here this line of applications and concentrate on the weighted inequalities, such as (1.1).

In analogy with the Gaussian case, it was recently shown in [12] that the generalized Cauchy distributions $\mu = \nu_\beta$ on $\mathbb{R}^n$, which have densities of the form $\frac{d\nu_\beta(x)}{dx} = \frac{1}{Z}\,(1 + |x|^2)^{-\beta}$, satisfy for $\beta > n$ the weighted Poincaré-type inequality

$$\mathrm{Var}_{\nu_\beta}(f) \leq C_\beta \int \big|\nabla f(x)\big|^2 \big(1 + |x|^2\big)\, d\nu_\beta(x). \tag{1.2}$$

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Moreover, one may choose the constants C_β behaving like $1/(2\beta)$ for large values of β. Since after rescaling of the coordinates the measures ν_β approximate γ_n, (1.2) may be used to recover the Gaussian Poincaré-type inequality. For related matters and different approaches, see [5, 6, 18, 20].

Apart from the problem on optimal constants, this example may further be generalized to include arbitrary convex probability measures with a fixed parameter of convexity. A probability measure μ on $\mathbb{R}^n$ is called κ-concave, where $-\infty \leq \kappa \leq +\infty$, if it satisfies the Brunn-Minkowski-type inequality

$$\mu\big(tA + (1-t)B\big) \geq \big[t\mu(A)^\kappa + (1-t)\mu(B)^\kappa\big]^{1/\kappa} \tag{1.3}$$

for all $t \in (0,1)$ and for all Borel measurable sets $A, B \subset \mathbb{R}^n$ with positive measure. Here $tA + (1-t)B = \{tx + (1-t)y : x \in A, y \in B\}$ stands for the Minkowski sum of the two sets. When $\kappa = 0$, the inequality (1.3) becomes

$$\mu\big(tA + (1-t)B\big) \geq \mu(A)^t \mu(B)^{1-t},$$

and we arrive at the notion of a log-concave measure, introduced by A. Prékopa (cf. [32, 37, 38]). When $\kappa = -\infty$, the right-hand side is understood as $\min\{\mu(A), \mu(B)\}$. The inequality (1.3) is getting stronger as the parameter κ is increasing, so in the case $\kappa = -\infty$ we obtain the largest class, whose members are called convex or hyperbolic probability measures. For general κ's, the family of κ-concave measures was introduced and studied by C. Borell [14, 15], cf. also [16].

A remarkable feature of this family is that many important geometric properties of κ-concave measures may be controlled by the parameter κ, only, and in essense do not depend on the dimension n (like the properties expressed in terms of Khinchin and dilation-type inequalities). This is in despite of the fact that the dimension appears in the density description of many κ-concave measures. Indeed, one may start with an arbitrary probability density $p(x) = \frac{d\mu(x)}{dx} = V(x)^{-\beta}$, where V is a positive convex function on an open supporting set $\Omega \subset \mathbb{R}^n$, with $\beta \geq n$, and then we obtain a κ-concave probability measure with a negative parameter $\kappa = -1/(\beta - n)$.

Let us return to the weighted type inequalities (1.1)–(1.2). In [12] it was also shown that any κ-concave probability measure μ on $\mathbb{R}^n$ satisfies

$$\mathrm{Var}_\mu(f) \leq C_\mu \int \big|\nabla f(x)\big|^2 \big(1 + |x|^2\big)\, d\mu(x),$$

thus, up to a constant with the same weight function as in the Cauchy case. Moreover, within universal factors, there is a stronger analytic form, which in turn may equivalently be described as an isoperimetric inequality of Cheeger's type.

The purpose of this note is to derive a more precise inequality, which would correctly reflect the behaviour of the weight function with respect to the parameter κ, especially when it is close to zero. To this task, introduce the geometric mean for the Euclidean norm under μ,

$$m_0 = \exp \int \log|x|\, d\mu(x).$$

It is finite for any κ-concave probability measure with finite κ and may be shown to be equivalent to the median of the Euclidean norm (up to factors, depending on κ).

The main statement of this work is the following theorem.

Theorem 1.1. *If μ is a κ-concave probability measure on $\mathbb{R}^n$, $-\infty < \kappa \leq 0$, for any locally Lipschitz function f on $\mathbb{R}^n$ we have*

$$\iint |f(x) - f(y)|\, d\mu(x) d\mu(y) \leq C_\kappa \int \big|\nabla f(x)\big| \big(m_0 - \kappa|x|\big)\, d\mu(x) \tag{1.4}$$

with constants C_κ that continuously depend on κ in the indicated range.

One may equivalently rephrase Theorem 1.1 as the following statement of isoperimetric flavor: For all non-empty Borel sets A and B in $\mathbb{R}^n$ located at distance $h = \mathrm{dist}(A, B) > 0$,

$$\mu(A)\,\mu(B) \,\leq\, \frac{C_\kappa}{2h} \int_{\mathbb{R}^n \setminus (A \cup B)} (m_0 - \kappa|x|)\, d\mu(x). \tag{1.5}$$

In particular, in the log-concave case ($\kappa = 0$) we arrive at

$$\mu(A)\,\mu(B) \,\leq\, \frac{C_0 m_0}{2h}\,(1 - \mu(A \cup B)), \tag{1.6}$$

where C_0 is a universal constant. This is one of the variants of the isoperimetric inequality of Cheeger's type

$$\mu^+(A) \,\geq\, \frac{2}{C_0 m_0}\, \mu(A)\, \big(1 - \mu(A)\big), \tag{1.7}$$

relating the μ-perimeter of the set $\mu^+(A) = \liminf_{h \to 0} \frac{\mu(A^h) - \mu(A)}{h}$ to the size $\mu(A)$ (where A^h denotes an h-neighbourhood of A).

For the uniform distributions on convex bodies (which correspond to the value $\kappa = 1/n$ in the hierarchy of convex measures), inequalities (1.6)-(1.7) were obtained by R. Kannan, L. Lovász and M. Simonovits in [27]. In fact, their localization approach carries over the class of general log-concave measures; cf. also [7] for a different approach. By a standard argument due to V. G. Maz'ya and J. A. Cheeger, (1.6)-(1.7) imply the usual Poincaré-type inequality

$$\mathrm{Var}_\mu(f) \leq C_0^2 m_0^2 \int \big|\nabla f(x)\big|^2 d\mu(x), \tag{1.8}$$

which in turn, up to a universal factor, may be shown to imply (1.6)-(1.7) in the class of log-concave probability measures, cf. [31]. Let us emphasize that in the log-concave case the geometric mean m_0 is equivalent to the L^1-norm $m_1 = \int |x|\, d\mu(x)$, or one may also take L^2-norm. When, however, κ is negative, m_1 might be infinite, and we need to involve other characteristics, such as m_0.

Thus, the inequalities (1.4)-(1.5) may be viewed as a natural extension of the Kannan-Lovász-Simonovits theorem to the family of κ-concave measures with negative finite κ's. Correspondingly, as an extension of (1.8) to this family we derive from Theorem 1.1:

Theorem 1.2. *If μ is a κ-concave probability measure on $\mathbb{R}^n$, $-\infty < \kappa \leq 0$, for any locally Lipschitz function f with finite μ-variance, we have*

$$\mathrm{Var}_\mu(f) \leq C_\kappa \int \big|\nabla f(x)\big|^2 \big(m_0^2 + \kappa^2 |x|^2\big)\, d\mu(x), \tag{1.9}$$

where the constants C_κ continuously depend on κ.

The paper is organized as follows. In Section 2 we recall basic facts, describing the class of κ-concave measures. Some additional results about dimension one are collected in Section 3. Here we treat general κ-concave measures as "nice" transformations of the so-called Pareto distributions on the line, which will allow us in Section 5 to reach a one-dimensional variant of Theorem 1.1. One of the ingredients in the argument is based on Khinchin-type inequalities for norms, which we consider separately in Section 4. In Section 6, which is rather general and where convexity does not play any special role, we discuss equivalent forms for the analytic inequality (1.4), including (1.5) and some other. (Two general statements are postponed to the Appendix). An important localization argument, which is used to extend these inequalities from the line to higher dimensions, is discussed in Section 7. Finally, in Section 8 we make final steps towards Theorems 1.1 and 1.2.

Contents

2. Characterization of κ-concave measures

A full characterization of κ-concave measures was given by C. Borell in [14, 15], cf. also [16]. Namely, any κ-concave probability measure is supported on some (relatively) open convex set $\Omega \subset \mathbb{R}^n$ and is absolutely continuous with respect to Lebesgue measure on Ω. Necessarily, $\kappa \leq 1/\dim(\Omega)$, and if Ω has dimension n, we have:

Proposition 2.1. *An absolutely continuous probability measure μ on $\mathbb{R}^n$ is κ-concave, where $-\infty \leq \kappa \leq 1/n$, if and only if μ is supported on an open convex set $\Omega \subset \mathbb{R}^n$, where it has a positive density p such that, for all $t \in (0,1)$ and $x, y \in \Omega$,*

$$p(tx + (1-t)y) \geq \left[\, tp(x)^{\kappa_n} + (1-t)p(y)^{\kappa_n}\right]^{1/\kappa_n}, \tag{2.1}$$

where $\kappa_n = \frac{\kappa}{1-n\kappa}$.

If κ is negative, one may represent the density in the form $p = V^{-\beta}$ with $\beta \geq n$, $\kappa = -1/(\beta - n)$, where V is an arbitrary positive convex function on Ω, satisfying the normalization condition $\int_\Omega V^{-\beta}\,dx = 1$.

The κ-mean function,

$$M_\kappa^{(t)}(a,b) = \left[ta^\kappa + (1-t)b^\kappa\right]^{1/\kappa}, \qquad a, b \geq 0,$$

which appears both in the Brunn-Minkowsi-type inequality (1.3) and in the density description (2.1), is understood as $a^t b^{1-t}$ for $\kappa = 0$, and as $\min\{a,b\}$ for $\kappa = -\infty$.

In dimension one there is another equivalent characterization. If an absolutely continuous probability measure μ on the real line $\mathbb{R}$ is supported on an open interval

(a,b), bounded or not, and has there a positive continuous density p, one may associate to it the function

$$I_\mu(t) = p(F^{-1}(t)), \quad 0 < t < 1,$$

where $F^{-1} : (a,b) \to (0,1)$ is the inverse to the distribution function $F(x) = \mu(a,x)$, $a < x < b$. Conversely, starting from a continuous positive function I on $(0,1)$, we obtain via the equality

$$F^{-1}(t) - F^{-1}(t_0) = \int_{t_0}^{t} \frac{ds}{I(s)}, \qquad 0 < t, t_0 < 1, \tag{2.2}$$

a probability measure μ with the distribution function F and the associated function $I_\mu = I$. (The measure is unique modulo shifts.)

With these notations and assumptions, we have (cf. [9]):

Proposition 2.2. *The probability measure μ is κ-concave, where $-\infty < \kappa < 1$, if and only if the function $I_\mu^{1/(1-\kappa)}$ is concave on $(0,1)$.*

This follows from Proposition 2.1 and the general identity

$$\kappa\, (I_\mu^{1/(1-\kappa)})'(F(x)) = (p(x)^{\kappa/(1-\kappa)})', \quad \kappa \neq 0.$$

Note, in the case $\kappa = 1$, the class of non-degenerate κ-concave measures coincides with the class of uniform distributions on bounded intervals.

3. Transforms of Pareto distributions

In this section we consider one-dimensional measures, only. In this case, Proposition 2.2 may be used to represent κ-concave measures as "nice" transforms of certain "standard" κ-concave measures. Note the concavity of $I_\mu^{1/(1-\kappa)}$ implies that the limit $I_\mu(0+) = \lim_{t\to 0+} I(t)$ exists, is finite, and the associated function admits a lower bound

$$I_\mu(t) \geq I_\mu(0+)\,(1-t)^{1-\kappa}, \qquad 0 < t < 1. \tag{3.1}$$

Here, an equality is attained for some probability distributions which play an important role in the class of κ-concave measures on the real line. Namely, given a finite $\kappa \leq 1$, introduce a probability measure μ_κ on the positive half-axis $(0,+\infty)$ with the distribution function

$$F_\kappa(x) = 1 - (1-\kappa x)^{1/\kappa}, \quad 0 < x < c_\kappa, \tag{3.2}$$

and the associated function $I_{\mu_\kappa}(t) = (1-t)^{1-\kappa}$. When $-\infty < \kappa < 0$, the measure μ_κ represents a Pareto distribution with parameter $\alpha = -1/\kappa$, which in the limit (or for $\kappa = 0$) becomes a one-sided exponential distribution with density $p(x) = e^{-x}$, $x > 0$. In these cases $c_\kappa = +\infty$. When $0 < \kappa \leq 1$, μ_κ is supported on the finite interval $(0, 1/\kappa)$, that is, $c_\kappa = 1/\kappa$.

Now, let μ be a probability measure, supported on an open interval (a,b), $-\infty \leq a < b \leq +\infty$, and having there a positive continuous density p. Consider the (unique) increasing map $T : (0, c_\kappa) \to (a,b)$, which pushes forward μ_κ to μ, i.e.,

$$T(x) = F^{-1}(F_\kappa(x)), \quad 0 < x < c_\kappa,$$

where $F^{-1} : (a,b) \to (0,1)$ is the inverse to the distribution function F of μ.

Proposition 3.1. *If μ is κ-concave with $-\infty < \kappa \leq 1$, and $a > -\infty$, then T is concave on $(0, c_\kappa)$ and has a Lipschitz seminorm*

$$\|T\|_{\mathrm{Lip}} \leq \frac{1}{p(a+)}.$$

Proof. In terms of the associated functions the property $\|T\|_{\mathrm{Lip}} \leq C$ is equivalent to $C\, I_\mu \geq I_{\mu_\kappa}$, which is indeed fulfilled with $C = 1/p(a+)$, according to (3.1) and since $I_\mu(0+) = p(a+)$.

Now, using the κ-concavity of μ, apply the definition (1.3) to the half-axes $A = (x, +\infty)$, $B = (y, +\infty)$ with arbitrary $x, y \in (a, b)$. Then, for all $t \in (0,1)$, $s = 1-t$,

$$1 - F(tx + sy) \geq M_\kappa^{(t)}(1 - F(x), 1 - F(y)). \tag{3.3}$$

Note that for the distribution F_κ, defined in (3.2), the above inequality turns into an equality. Therefore, representing $F = F_\kappa(T^{-1})$, where $T^{-1} : (a,b) \to (0, c_\kappa)$ is the inverse map, we obtain from (3.3) that

$$\begin{aligned} 1 - F_\kappa\big(T^{-1}(tx + sy)\big) &\geq M_\kappa^{(t)}\big(1 - F_\kappa(T^{-1}(x)), 1 - F_\kappa(T^{-1}(y))\big) \\ &= 1 - F_\kappa\big(tT^{-1}(x) + sT^{-1}(y)\big). \end{aligned}$$

Hence, $T^{-1}(tx + sy) \leq tT^{-1}(x) + sT^{-1}(y)$, that is, T^{-1} is convex or T is concave. Proposition 3.1 follows. $\square$

There is another useful variant of Proposition 3.1 involving a median of the distribution. By the concavity of $I_\mu^{1/(1-\kappa)}$, we also have

$$I_\mu(t) \geq 2^{1-\kappa}\, I_\mu(1/2)\, (\min\{t, 1-t\})^{1-\kappa}, \qquad 0 < t < 1. \tag{3.4}$$

Here, up to the factor, an equality is attained for a symmetric probability distribution ν_κ with the associated function $I_{\nu_\kappa}(t) = (\min\{t, 1-t\})^{1-\kappa}$. Its distribution function is given by

$$\nu_\kappa(x, +\infty) = \frac{1}{(2^{-\kappa} - \kappa x)^{-1/\kappa}}, \quad x \geq 0, \tag{3.5}$$

so, when $-\infty < \kappa < 0$, ν_κ may be viewed as a symmetrized Pareto distribution, which in the limit (or for $\kappa = 0$) becomes a two-sided exponential distribution with density $p(x) = \frac{1}{2}\, e^{-|x|}$. Note ν_κ is supported on the finite interval $(-\frac{2^{-\kappa}}{\kappa}, \frac{2^{-\kappa}}{\kappa})$ when $0 < \kappa \leq 1$.

Since $I(1/2) = p(m)$, where m is the median of μ, we have (with a similar argument as in the proof of Proposition 3.1):

Proposition 3.2. *Let μ be a non-degenerate κ-concave probability measure on the real line with density p and median m $(-\infty < \kappa \leq 1)$. Then μ represents the image of the measure ν_κ under a non-decreasing Lipschitz map $T : \mathbb{R} \to \mathbb{R}$ with*

$$\|T\|_{\mathrm{Lip}} \leq \frac{1}{2^{1-\kappa} p(m)}.$$

To make the obtained bounds on the Lipschitz seminorm practical, it is natural to relate the quantity $p(m)$ to integral characteristics of random variables with distribution μ. For a random variable X introduce L^q-norms $\|X\|_q = (\mathbb{E}\,|X|^q)^{1/q}$, including the geometric mean

$$\|X\|_0 = \exp\, \mathbb{E} \log |X|.$$

If X has a κ-concave distribution for some $\kappa < 0$, then $\|X\|_q < +\infty$ for all $q < -1/\kappa$, so $\|X\|_0 = \lim_{q \to 0} \|X\|_q$ is always finite (cf. [9, 25]).

Proposition 3.3. *There are positive continuous functions $C_1(\kappa)$ and $C_2(\kappa)$ in the range $-\infty < \kappa \le 1$ with the following property. For any random variable X, having a non-degenerate κ-concave distribution with density p and median m,*

$$C_1(\kappa)\, \|X-m\|_0 \le \frac{1}{p(m)} \le C_2(\kappa)\, \|X-m\|_0. \tag{3.6}$$

For a proof first note that, by the concavity of $I_\mu^{1/(1-\kappa)}$, we immediately obtain:

Lemma 3.4. *If a non-degenerate probability measure μ on the real line with density p and median m is κ-concave, $-\infty < \kappa \le 1$, then*

$$2^{\kappa-1} \sup_x p(x) \le p(m) \le \sup_x p(x).$$

Proof of Proposition 3.3. If F is the distribution function of X with inverse F^{-1} and the associated function $I(t) = p(F^{-1}(t))$, then $X-m$ has the same distribution as the function

$$F^{-1}(t) - F^{-1}(1/2) = \int_{1/2}^t \frac{ds}{I(s)}$$

has under the Lebesgue measure on (0,1). In particular, for any $q > 0$,

$$\|X-m\|_q^q = \int_0^1 \bigg| \int_{1/2}^t \frac{ds}{I(s)}\bigg|^q\, dt. \tag{3.7}$$

Using $I(s) \le \sup_x p(x)$, we get $|\int_{1/2}^t \frac{ds}{I(s)}| \ge \frac{1}{\sup p}\,|t - \frac{1}{2}|$ and

$$\|X-m\|_q^q \ge \frac{1}{(\sup_x p(x))^q} \int_0^1 \Big|t - \frac{1}{2}\Big|^q\, dt = \frac{1}{(\sup_x p(x))^q}\, \frac{1}{2^q\,(q+1)}.$$

With Lemma 3.4, this gives $\|X-m\|_q \ge \frac{1}{p(m)}\, \frac{1}{2^{2-\kappa}\,(q+1)^{1/q}}$. Letting $q \to 0$, we arrive at the second inequality in (3.6) with $C_2(\kappa) = 2^{2-\kappa}\, e$.

For the converse bound, one may apply Proposition 3.2 or directly (3.4) with (3.7), which yield

$$\|X-m\|_q \le \frac{1}{2^{1-\kappa}\, p(m)}\, \|X_\kappa\|_q,$$

where X_κ is a random variable with the distribution ν_κ introduced in (3.5). Hence, one may take $C_1(\kappa) = 2^{1-\kappa}\, \|X_\kappa\|_0$. □

However, in further applications it will be more convenient to bound $1/p(m)$ in terms of $\|X\|_0$, rather than to work with $\|X-m\|_0$. To get a desired estimate, one may use a very general principle concerning the integrals of the form

$$L_\mu = \int \Phi(|x|)\, d\mu(x),$$

where Φ is a given non-decreasing function on $[0,+\infty)$. Namely, in the class of all absolutely continuous probability measures μ on $\mathbb{R}^n$ with densities p such that $\sup_x p(x) \le 1$, the functional L_μ is minimized, when μ represents a uniform distribution λ on the Euclidean ball $B(0,R)$ with center at the origin and volume one (so that $\omega_n R^n = 1$, where ω_n is the volume of the unit ball). When $\Phi(r) = r^2$, this observation was first made by D. Hensley [26] (who considered log-concave densities) and then was stated in the general situation by K. Ball [2].

For a simple argument, let us note that L_μ is linear with respect to Φ, so one may assume $\Phi = 1_{(r,\infty)}$, the indicator function of a half-axis. Then the property $L_\mu \geq L_\lambda$ reads as

$$\mu\{|x| \leq r\} \ \leq \ \lambda\{|x| \leq r\} \ = \ \begin{cases} \omega_n r^n, \text{ for } \ 0 \leq r \leq R, \\ \ \ 1, \quad \text{for } \ r > R. \end{cases}$$

This inequality is automatically fulfilled, when $r > R$. In the other case, due to the assumption $p \leq 1$, we have

$$\mu\{|x| \leq r\} = \int_{\{|x| \leq r\}} p(x)\,dx \leq \int_{\{|x| \leq r\}} dx = \mathrm{vol}_n(B(0,r)) = \lambda\{|x| \leq r\},$$

which is the statement.

In particular, subject to the condition $\sup_x p(x) = 1$, we have that, for any $q > 0$, the one-dimensional integral $\int_{-\infty}^{+\infty} |x|^q \, d\mu(x)$ is minimized for the uniform distribution on $(-1/2, 1/2)$, and the minimum is equal to $\frac{1}{2^q\,(1+q)}$. Equivalently, if X is a random variable with density p,

$$\frac{1}{\sup_x p(x)} \ \leq \ 2\,(1+q)^{1/q}\,\|X\|_q.$$

In the limit case $q = 0$, we also have $\frac{1}{\sup_x p(x)} \ \leq \ 2e\,\|X\|_0$. Recalling Lemma 3.4, we arrive at:

Proposition 3.5. *For any random variable X, having a non-degenerate κ-concave distribution with density p and median m,*

$$\frac{1}{p(m)} \ \leq \ 2^{2-\kappa} e\,\|X\|_0.$$

Note the constant here is the same as the constant $C_2(\kappa)$ from Proposition 3.3.

4. Median and geometric mean of norms

To proceed, we need reasonable integral estimates for the median. In fact, for convex measures rather sharp and at the same time general bounds are available, which are treated in the scheme of the so-called dilation-type inequalities. Special cases and for the class of log-concave measures dilation-type inequalities were considered in many works, starting in [14] (cf. for an account [36, 8]). The class of κ-concave measures was, however, considered only recently. One particular case in such inequalities corresponds to the dilation of symmetric convex bodies, and we state it below ([13], cf. also [21]).

Proposition 4.1. *Given a κ-concave probability measure μ on $\mathbb{R}^n$, $-\infty < \kappa \leq 1$, for any symmetric, convex set B in $\mathbb{R}^n$ and for all $h > 1$,*

$$1 - \mu(B) \ \geq \ \left[\frac{2}{h+1}\ (1 - \mu(hB))^\kappa + \frac{h-1}{h+1}\right]^{1/\kappa}. \tag{4.1}$$

When $\kappa = 0$, the above reads as $1 - \mu(B) \geq (1 - \mu(hB))^{2/(h+1)}$, or equivalently,

$$1 - \mu(hB) \leq (1 - \mu(B))^{(h+1)/2},$$

which is due to L. Lovász and M. Simonovits [33] in case of Euclidean balls B. O. Guédon [25] extended this inequality to general convex bodies and also found a precise relation in the case $\kappa > 0$. Namely, then (4.1) is solved in terms of $1-\mu(hB)$ as

$$1 - \mu(hB) \ \leq \ \max{}^{1/\kappa}\Big\{\frac{h+1}{2}\,(1-\mu(B))^\kappa - \frac{h-1}{2},\, 0\Big\}.$$

We are mostly interested in the range $\kappa < 0$, when (4.1) yields

$$1 - \mu(hB) \ \leq \ \bigg[\,\frac{h+1}{2}\,(1-\mu(B))^\kappa - \frac{h-1}{2}\,\bigg]^{1/\kappa}. \tag{4.2}$$

Let X be a random vector in $\mathbb{R}^n$ with distribution μ. If $\mathbb{R}^n$ is equipped with a norm $\|\cdot\|$, Proposition 4.1 may be used to study integrability properties of the random variable $\|X\|$. This way one can reach various Khinchin-type inequalities for the L^q-norms $\|X\|_q = (\mathbb{E}\,\|X\|^q)^{1/q}$ under κ-concave measures, including the geometric mean

$$\|X\|_0 = \exp\,\mathbb{E}\log\|X\|.$$

In particular, this quantity may be related to the median $m = m(\|X\|)$ of $\|X\|$, which is defined in the usual way as a number such that

$$\mathbb{P}\{\|X\| \leq m\} = \frac{1}{2}.$$

More precisely, what we need is:

Corollary 4.2. *If X has a non-degenerate κ-concave distribution on $\mathbb{R}^n$, then*

$$C_1(\kappa)\,\|X\|_0 \leq m(\|X\|) \leq C_2(\kappa)\,\|X\|_0 \tag{4.3}$$

for some positive continuous functions C_1, C_2 of κ in the range $-\infty < \kappa \leq 1$.

Proof. It is enough to consider the values $\kappa < 0$ and to obtain (4.3) with some continuous functions $C_{1,2}$ having finite limits $C_{1,2}(0-)$ as $\kappa \to 0$ (and then one may put $C_{1,2}(\kappa) = C_{1,2}(0-)$ for $\kappa > 0$).

If $\kappa < 0$, applying (4.2) to the convex set $B = \{x \in \mathbb{R}^n : \|x\| \leq m\}$, we obtain a large deviation inequality

$$\mathbb{P}\{|X| > mh\} \ \leq \ \bigg[\,\frac{h+1}{2}\,2^{-\kappa} - \frac{h-1}{2}\,\bigg]^{1/\kappa}, \quad h \geq 1. \tag{4.4}$$

Thus, $\mathbb{P}\{\|X\| > mh\} = O(h^{1/\kappa})$, as $h \to +\infty$, a property mentioned by C. Borell in [14]. To get a more precise information, we replace $h = 1 + x$ with $x \geq 0$ and notice that

$$\frac{h+1}{2}\,2^{-\kappa} - \frac{h-1}{2} \ = \ 2^{-\kappa} + \frac{2^{-\kappa}-1}{2}\,x \ \geq \ 2^{-\kappa} + (-\kappa)\,\frac{\log 2}{2}\,x,$$

so that, by (4.4),

$$\mathbb{P}\{\|X\| > m(1+x)\} \ \leq \ \bigg[\,2^{-\kappa} + (-\kappa)\,\frac{\log 2}{2}\,x\,\bigg]^{1/\kappa}, \quad x \geq 0.$$

Here, up to the factor $c = \frac{\log 2}{2}$, the right-hand side may be recognized as the tail function of the symmetrized Pareto distribution ν_κ, cf. (3.5). In other words, if X_κ is a random variable with distribution ν_κ,

$$\mathbb{P}\{\|X\| > m(1+x)\} \ \leq \ \frac{1}{2}\,\mathbb{P}\{|X_\kappa| > cx\}, \quad x \geq 0.$$

Therefore, introducing the distribution functions $F(h) = \mathbb{P}\{\|X\| \leq mh\}$ and $G(x) = \mathbb{P}\{|X_\kappa| \leq cx\}$, we have, for all $q > 0$,

$$\begin{aligned}\mathbb{E}\left(\frac{\|X\|}{m}\right)^q &= \int_0^{+\infty} (1 - F(h))\, dh^q \leq 1 + \int_1^{+\infty} (1 - F(h))\, dh^q \\ &\leq 1 + \frac{1}{2}\int_0^{+\infty} (1 - G(x))\, d(1+x)^q \\ &= \frac{1}{2} + \frac{1}{2}\int_0^{+\infty} (1+x)^q\, dG(x) \ = \ \frac{1}{2} + \frac{1}{2}\,\mathbb{E}\left(1 + \frac{|X_\kappa|}{c}\right)^q.\end{aligned}$$

This gives

$$\|X\|_q \ \leq \ m\left(\frac{1}{2} + \frac{1}{2}\left\|1 + \frac{|X_\kappa|}{c}\right\|_q^q\right)^{1/q} \tag{4.5}$$

with finite right-hand side, as long as $q < -1/\kappa$. Since $\|1 + \frac{|X_\kappa|}{c}\|_q \to \|1 + \frac{|X_\kappa|}{c}\|_0$, as $q \to 0$ (with a finite limit), we get that

$$\begin{aligned}\|X\|_0 &\leq m\ \lim_{q\to 0}\left(\frac{1}{2} + \frac{1}{2}\left\|1 + \frac{|X_\kappa|}{c}\right\|_q^q\right)^{1/q} \\ &= m\ \lim_{q\to 0}\left(\frac{1}{2} + \frac{1}{2}\left\|1 + \frac{|X_\kappa|}{c}\right\|_0^q\right)^{1/q} \ = \ m\left(\left\|1 + \frac{|X_\kappa|}{c}\right\|_0\right)^{1/2}.\end{aligned}$$

Thus, the first inequality in (4.3) holds with constant, defined by

$$\frac{1}{C_1^2(\kappa)} = \left\|1 + \frac{|X_\kappa|}{c}\right\|_0.$$

The second inequality should be based on a bound for "small ball probabilities". In fact, such a bound can be derived from the same dilation-type inequality (4.2). Namely, taking there $B = \{x \in \mathbb{R}^n : \|x\| \leq m\varepsilon\}$ with fixed $\varepsilon \in (0,1]$ and applying it to $h = 1/\varepsilon$, we may write (4.2) as

$$\frac{1}{2} \ \leq \ \left[\frac{1+\varepsilon}{2\varepsilon}\ \mathbb{P}\{\|X\| > m\varepsilon\}^\kappa - \frac{1-\varepsilon}{2\varepsilon}\right]^{1/\kappa}$$

and solve as

$$\mathbb{P}\{\|X\| \leq m\varepsilon\} \ \leq \ 1 - \left[1 + (2^{-\kappa} - 1)\,\frac{2\varepsilon}{1+\varepsilon}\right]^{1/\kappa}. \tag{4.6}$$

To simplify, put $\alpha = -1/\kappa$, $x = (2^{-\kappa} - 1)\,\frac{2\varepsilon}{1+\varepsilon}$, and consider the function $\phi(x) = 1 - (1+x)^{-\alpha}$ which appears on the right-hand side of (4.6). Since this function is concave in $x > -1$, we have $\phi(x) \leq \phi(0) + \phi'(0)x = \alpha x$, so

$$\mathbb{P}\{\|X\| \leq m\varepsilon\} \ \leq \ \frac{2^{-\kappa} - 1}{-\kappa}\ \frac{2\varepsilon}{1+\varepsilon}, \qquad 0 < \varepsilon \leq 1. \tag{4.7}$$

Now, applying (4.7),

$$\begin{aligned}\mathbb{E}\log\frac{\|X\|}{m} &= \int_0^{+\infty} \log\varepsilon\ d\,\mathbb{P}\{\|X\| \leq m\varepsilon\} \\ &\geq \int_0^1 \log\varepsilon\ d\,\mathbb{P}\{\|X\| \leq m\varepsilon\} \ = \ -\int_0^1 \frac{\mathbb{P}\{\|X\| \leq m\varepsilon\}}{\varepsilon}\ d\varepsilon \\ &\geq \frac{2^{-\kappa} - 1}{\kappa}\int_0^1 \frac{2\,d\varepsilon}{1+\varepsilon} \ = \ \frac{2^{-\kappa} - 1}{\kappa}\ \log 4.\end{aligned}$$

Therefore, $\frac{\|X\|_0}{m} \geq 4^{\frac{2^{-\kappa}-1}{\kappa}}$, that is, the second inequality in (4.3) holds with constant

$$C_2(\kappa) = 4^{\frac{2^{-\kappa}-1}{-\kappa}}.$$

Note that $C_2(0) = \lim_{\kappa \to 0} C_2(\kappa) = 4^{\log 2}$.

A combination of the inequality (4.5), which we obtained in the proof of the lower bound on the median $m = m(\|X\|)$ in Corollary 4.2, together with the upper bound immediately leads to a Khinchine-type inequality for L^q-norms. □

Corollary 4.3. *If X has a non-degenerate κ-concave distribution on $\mathbb{R}^n$ with $-\infty < \kappa < 0$, then for all $q < -1/\kappa$,*

$$\|X\|_q \leq C(\kappa, q)\, \|X\|_0,$$

where C continuously depends on (κ, q).

For example, when $q = 1$ and $\kappa = -1/2$, we have with a universal constant C

$$\|X\|_1 \leq C\, \|X\|_0. \tag{4.8}$$

5. Weighted Cheeger-type inequalities on the line

It is time to explain how to get functional forms for Cheeger-type (isoperimetric) inequalities with weight on the real line. Although there is a full characterization of probability measures that satisfy Hardy-type inequalities with weight (cf. [34, 35]), still we would be lead to questions on the dependence of the constants in such inequalities on the involved measures.

Instead, let us start with the standard Pareto distributions μ_κ, $\kappa < 0$, which we discussed in Section 2, cf. (3.2). These measures are concentrated on the positive half-axis $(0, +\infty)$ and have the tail function

$$1 - F_\kappa(x) = (1 - \kappa x)^{1/\kappa}, \quad x \geq 0,$$

that is, $d\mu_\kappa(x) = -(1 - \kappa x)^{1/\kappa - 1}\, dx$.

Given a smooth function f on $[0, +\infty)$ with $f(0) = 0$ (and having a compactly supported derivative), we get, integrating by parts, that

$$\int f(x)\, d\mu_\kappa(x) = \int_0^{+\infty} f'(x)\, (1 - F_\kappa(x))\, dx = \int_0^{+\infty} f'(x)\, (1 - \kappa x)\, d\mu_\kappa(x).$$

This identity easily yields the inequality

$$\int |f(x)|\, d\mu_\kappa(x) \leq \int_0^{+\infty} |f'(x)|\, (1 - \kappa x)\, d\mu_\kappa(x), \tag{5.1}$$

which is true for all locally Lipschitz f on $[0, +\infty)$ with $f(0) = 0$ (and where f' is understood as the Radon-Nikodym derivative).

Thus, we arrived in (5.1) at a functional Cheeger-type inequality for μ_κ with weight $w(x) = 1 - \kappa x$. The general κ-concave case may be treated via (5.1) using transforms of the measures μ_κ.

Proposition 5.1. *Let μ be a non-degenerate κ-concave probability measure on $\mathbb{R}$ with density p and median m, $-\infty < \kappa < 0$. For any locally Lipschitz function f on $\mathbb{R}$,*

$$\int |f - f(m)|\, d\mu \leq \int |f'(x)| \Big(\frac{1}{2p(m)} - \kappa |x - m|\Big)\, d\mu(x). \tag{5.2}$$

Since the functional $c \rightarrow \int |f - c| \, d\mu$ is minimized for (any) median $c = m(f)$ of f under μ, the above inequality yields

$$\int |f - m(f)| \, d\mu \, \leq \, \int |f'(x)| \Big(\frac{1}{2p(m)} - \kappa |x - m| \Big) \, d\mu(x). \tag{5.3}$$

The log-concave case may also be included in this statement by letting $\kappa \rightarrow 0$, and then we arrive at the usual Cheeger-type inequality without weights,

$$\int |f - m(f)| \, d\mu \, \leq \, \frac{1}{2p(m)} \int |f'| \, d\mu.$$

In other words, Cheeger's isoperimetric constant $Is(\mu)$ of μ is bounded from below by $2p(m)$. In fact, the above inequality is optimal, i.e., $Is(\mu)$ is equal to $2p(m)$, cf. [7], Section 4.

Proof of Proposition 5.1. Shifting the measure μ (if necessary), we may assume $m = 0$, as well as $f(0) = 0$. Let (a,b) be a supporting interval of μ, so that $-\infty \leq a < 0 < b \leq +\infty$, and denote by μ_+ the normalized restriction of μ to $(0,b)$. Thus, it has the density $p_+(x) = 2p(x)$, $x \in (0,b)$.

Introduce the increasing map $T : (0,+\infty) \rightarrow (0,b)$, which pushes forward the Pareto distribution μ_κ to μ_+. Note $T(0+) = 0$, so we may apply (5.2) to the function $f(T)$, which gives

$$\int_0^b |f| \, d\mu_+ \leq \int_0^{+\infty} |f'(T(x))| \, T'(x) \, (1 - \kappa x) \, d\mu_\kappa(x). \tag{5.4}$$

Here the expression $T'(x)\,(1 - \kappa x) = T'(x) - \kappa \, T'(x) \, x$ on the right-hand side can be estimated from above with the help of Proposition 3.1. For the first term, we use the bound $\|T\|_{\rm Lip} \leq 1/p_+(0+)$, and since $p_+(0+) = 2p(0)$, we get $T' \leq 1/(2p(0))$. To bound the second term, we use the concavity of the function T, which implies that $T'(x)x \leq T(x)$. Indeed, whenever $x > y > 0$, we may write $T(x) - T(y) \geq T'(x)(x - y)$ and then let $y \rightarrow 0$. Being combined, the two bounds give

$$T'(x)\,(1 - \kappa x) \leq \frac{1}{2p(0)} - \kappa T(x),$$

so by (5.4), after the change $y = T(x)$ we may return on its right-hand side to the measure μ_+ and arrive at

$$\int_0^b |f| \, d\mu_+ \leq \int_0^b |f'(y)| \, \Big(\frac{1}{2p(0)} - \kappa y \Big) \, d\mu_+(y).$$

With a similar inequality for the measure μ_-, the normalized restriction of μ to $(a,0)$, we finally obtain that

$$\int_a^b |f| \, d\mu \leq \int_a^b |f'(y)| \, \Big(\frac{1}{2p(0)} - \kappa |y| \Big) \, d\mu(y).$$

Proposition 5.1 is proved. $\square$

To further estimate the right-hand side of (5.2)-(5.3) in terms of integral characteristics of μ, one may just write $|x - m| \leq |x| + |m|$ and bound $|m|$ by the μ-median

$m(|x|)$ of the Euclidean distance. Next both $m(|x|)$ and $1/p(m)$ may be bounded from above by the geometric mean of the Euclidean distance,

$$m_0 = \exp \int \log |x| \, d\mu(x),$$

in accordance with Corollary 4.2 (the second inequality) and Proposition 3.5.

Corollary 5.2. *Given a non-degenerate κ-concave probability measure μ on $\mathbb{R}$ with $-\infty < \kappa \leq 0$, for any locally Lipschitz function f on $\mathbb{R}$ with μ-median $m(f)$,*

$$\int |f - m(f)| \, d\mu \leq \int |f'(x)| \, (C_\kappa \, m_0 - \kappa |x|) \, d\mu(x), \tag{5.5}$$

where C_κ is a continuous function of κ in the indicated range.

The assumption that μ is non-degenerate is not essential: If μ is concentrated at a point, say a, then $m_0 = a$, and (5.5) is immediate (the left-hand side is vanishing).

Using an elementary bound $\int |f - m(f)| \, d\mu \geq \frac{1}{2} \int |f - \int f \, d\mu| \, d\mu$, we may obtain from (5.5) an equivalent (within a factor of 2) analytic inequality of Cheeger-type

$$\frac{1}{2} \int \left| f - \int f \, d\mu \right| d\mu \leq \int |f'(x)| \, (C_\kappa \, m_0 - \kappa |x|) \, d\mu(x).$$

It should be understood in the standard sense: Any locally Lipschitz function f on $\mathbb{R}$, such that the right-hand side is finite, is integrable with respect to μ, and the inequality holds true.

As another variant, which does not require any integrability assumption, we may also write

$$\frac{1}{2} \iint |f(x) - f(y)| \, d\mu(x) d\mu(y) \leq \int |f'(x)| \, (C_\kappa \, m_0 - \kappa |x|) \, d\mu(x). \tag{5.6}$$

The form (5.6) is better adapted for multidimensional extensions. However, first we need to recall a family of geometric inequalities that may be used in place of (5.6).

6. Isoperimetric inequalities of Cheeger's type

Weighted analytic inequalities like (5.5)-(5.6) may equivalently, at least within universal factors, be stated on sets in a very general setting, say, on abstract metric spaces. However, here we restrict ourselves to the Euclidean space $\mathbb{R}^n$. In this section, we recall basic arguments, which lead to the equivalence between isoperimetric and analytic inequalities of Cheeger-type with weight. (For spaces with finite measure such statements are usually considered about inequalities without weight, cf. e.g. [19, 40], Theorem 1; [28, 10], Theorems 1.1-1.2).

For definiteness, assume we are given a non-negative Borel measurable function w on $\mathbb{R}^n$, not identically zero (called a weight function). To every Borel probability measure μ on $\mathbb{R}^n$ we associate a Borel measure, μ_w, which is absolutely continuous with respect to μ with density w, i.e.,

$$\mu_w(A) = \int_A w(x) \, d\mu(x), \quad A \subset \mathbb{R}^n \ \ \text{(Borel)}.$$

We assume, although this is not essential for most statements, that the function w is locally integrable with respect to μ. Equivalently, the measure μ_w is supposed to be finite on compact subsets of the space.

For any Borel set A in $\mathbb{R}^n$, one may define its (outer Minkowski) μ_w-perimeter

$$\mu_w^+(A) \, = \, \liminf_{\varepsilon \downarrow 0} \frac{\mu_w(A^\varepsilon \setminus A)}{\varepsilon},$$

where $A^\varepsilon = \{x \in \mathbb{R}^n : \exists y \in A, |x-y| < \varepsilon\}$ denotes an open Euclidean ε-neighbourhood of A. It is natural to restrict this definiton to those A's that have finite measure $\mu_w(A)$, and then

$$\mu_w^+(A) \, = \, \liminf_{\varepsilon \downarrow 0} \frac{\mu_w(A^\varepsilon) - \mu(A)}{\varepsilon}.$$

Recall, for example, that the classical isoperimetric inequality for the Lebesgue measure in $\mathbb{R}^n$ requires finiteness of the measure of a set.

If f is a locally Lipschitz function f on $\mathbb{R}^n$, its generalized modulus of the gradient is defined as the (finite Borel measurable) function

$$|\nabla f(x)| = \limsup_{y \to x} \frac{|f(x) - f(y)|}{|x-y|}, \qquad x \in \mathbb{R}^n.$$

Clearly, when the function is differentiable, we arrive at the usual definition.

We will say that f is μ_w-finite, if

$$\mu_w\{|f| > t\} < +\infty \quad \text{for all } t > 0.$$

For example, any μ_w-integrable function is μ_w-finite. This definition may be used in the setting of an abstract measure space (and indeed this property is essential in the Theory of Lebesgue integration over infinite measures). Note that if a measure ν is finite, then any measurable function on that space is ν-finite.

Proposition 6.1. *Given a Borel probability measure μ on $\mathbb{R}^n$, the following properties are equivalent:*

a) For any locally Lipschitz μ_w-finite function f on $\mathbb{R}^n$,

$$\int |\nabla f(x)| \, w(x) \, d\mu(x) \, \geq \, \frac{1}{2} \iint |f(x) - f(y)| \, d\mu(x) \, d\mu(y) \, ; \tag{6.1}$$

b) For any closed set A in $\mathbb{R}^n$ of finite measure $\mu_w(A)$, we have

$$\mu_w^+(A) \, \geq \, \mu(A)(1 - \mu(A)) \, ; \tag{6.2}$$

c) For any Borel set A in $\mathbb{R}^n$ of finite measure $\mu_w(A)$ and for any $h > 0$,

$$\frac{\mu_w(A^h) - \mu_w(A)}{h} \, \geq \, \mu(A)\big(1 - \mu(A^h)\big) \, ; \tag{6.3}$$

d) For all non-empty Borel sets $A, B \subset \mathbb{R}^n$ at distance $h > 0$, with finite $\mu_w(A)$,

$$\frac{1}{h} \int_{\mathbb{R}^n \setminus (A \cup B)} w \, d\mu \, \geq \, \mu(A) \, \mu(B) \, ; \tag{6.4}$$

e) (6.4) *holds true for all non-empty compact sets $A, B \subset \mathbb{R}^n$ at distance $h > 0$.*

Let us remind that the distance functional between non-empty sets is defined in the usual way as

$$h \,=\, \mathrm{dist}(A,B) \,=\, \inf_{x\in A,y\in B} |x-y|.$$

The geometric inequality (6.4) will be used for extensions of analytic inequalities with weight, such as (6.1), from the line to higher dimensions.

Remark 6.2. It is not clear if one may remove from the property a) the assumption that f is μ_w-finite, or equivalently, the assumption that $\mu_w(A)$ is finite in the property d). However, in some cases this can easily be done. For example, when the measure μ is compactly supported or can be approximated by compactly supported measures that satisfy a similar property such as (6.1) with a common (or asymptotically common) weight function, one may write this inequality for such measures without any assumption on the function f, and then in the limit we obtain (6.1) for μ in the class of all locally Lipschitz f. In particular, this argument may be applied to the family of κ-concave measures μ.

Remark 6.3. If $w(x) > 0$ on the support of μ, then μ is absolutely continuous with respect to μ_w, and as a result, the isoperimetric-type inequality (6.2) can be extended from the class of all closed sets to the class of all Borel sets $A \subset \mathbb{R}^n$ with finite $\mu_w(A)$.

Indeed, every A^ε contains $\mathrm{clos}(A)$, the closure of A. So, if $\mu_w(\mathrm{clos}(A)) > \mu_w(A)$, then $\mu_w^+(A) = +\infty$, and (6.2) is immediate. In case $\mu_w(\mathrm{clos}(A)) = \mu_w(A)$, we have that $\mu(\mathrm{clos}(A)) = \mu(A)$. This implies $\mu_w^+(\mathrm{clos}(A)) = \mu_w^+(A)$, and the inequality (6.2), being applied to the closed set $\mathrm{clos}(A)$, yields the same inequality for A.

Remark 6.4. It is not clear how optimal the inequalities (6.3)-(6.4) are. When the weight function w is constant, the isoperimetric-type inequality (6.2) takes the form

$$\mu^+(A) \,\geq\, c\mu(A)(1-\mu(A)).$$

As is known (cf. e.g. [10]), it may be "integrated" or "iterated" with respect to the parameter $h > 0$ to yield an equivalent relation

$$\mu(A^h) \,\geq\, \frac{p}{p + (1-p)\, e^{-ch}}$$

in the class of all Borel sets A in $\mathbb{R}^n$ of measure $p = \mu(A)$. Equality is attained for all $h > 0$ and $p \in (0,1)$ simulateneoulsy, when μ is the so-called logistic distribution on the real line, and when A is a half-axis.

Proof of Proposition 6.1. We involve in the proof a somewhat weakened variant of the property b):

b') For any closed set A in $\mathbb{R}^n$, such that $\mu_w(A^\varepsilon) < +\infty$ for some $\varepsilon > 0$, the weighted isoperimetric inequality (6.2) holds true.

We start with the equivalence between a) and b').

$a) \Rightarrow b'$): Assume $\mu_w(A^\varepsilon) < +\infty$ for some $\varepsilon > 0$. By Lemma 9.1 of the Appendix, there is a sequence of Lipschitz μ_w-finite functions $f_\ell : \mathbb{R}^n \to [0,1]$, such that

$$\limsup_{\ell\to\infty} \int |\nabla f_\ell|\, d\mu_w \leq \mu_w^+(A)$$

and $f_\ell \to 1_A$ pointwise. The latter yields

$$\lim_{\ell\to\infty} \iint |f_\ell(x) - f_\ell(y)|\, d\mu(x)\, d\mu(y) = 2\,\mu(A)(1-\mu(A)).$$

Hence, applying (6.1) to the the functions f_ℓ, in the limit we arrive at (6.2).

$b')\Rightarrow a)$: First assume f is Lipschitz and μ_w-finite. By Lemma 9.2, applied to the measure $\nu=\mu_w$, if the function

$$D_r f(x) \ = \sup_{0<|x-y|<r} \frac{|f(x)-f(y)|}{|x-y|}, \qquad x\in\mathbb{R}^n,$$

is μ_w-integrable for some $r>0$, then

$$\int_{\mathbb{R}^n} |\nabla f|\,d\mu_w \ \geq \ \int_{-\infty}^0 \mu_w^+\{f\leq t\}\,dt + \int_0^{+\infty} \mu_w^+\{f\geq t\}\,dt. \tag{6.5}$$

Note the integrals on the right-hand side do not depend on whether we use strict or non-strict inequalities in the integrands.

For $t>0$, the sets $A_t=\{f\geq t\}$ are closed, and by the Lipschitz property, $A^\varepsilon\subset\{f\geq t-M\varepsilon\}$, where $M=\|f\|_{\rm Lip}$. Hence, for each $t>0$, $\mu_w(A_t^\varepsilon)$ is finite for sufficiently small $\varepsilon>0$. Thus the assumption in property $b')$ is fulfilled, so that (6.2) may be applied to these sets:

$$\mu_w^+\{f\geq t\} \ \geq \ F(t-)(1-F(t-)), \quad \text{for any } t>0,$$

where $F(t)=\mu\{f\leq t\}$ is the distribution function of f under μ. Similarly,

$$\mu_w^+\{f\leq t\} \ \geq \ F(t)(1-F(t)), \quad \text{for any } t<0.$$

Hence, (6.5) yields

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla f|\,d\mu_w &\geq \int_{-\infty}^{+\infty} F(t)(1-F(t))\,dt\\ &= \frac{1}{2}\iint |f(x)-f(y)|\,d\mu(x)\,d\mu(y),\end{aligned} \tag{6.6}$$

that is, the desired inequality (6.1).

To remove the assumption that f is Lipschitz and $D_r f$ is μ_w-integrable, first assume f is μ_w-integrable. Note that, since f is locally Lipschitz, it has a finite Lipschitz seminorm on every ball in $\mathbb{R}^n$, so for any $r>0$, $D_r f$ is bounded on balls. Consider the Lipschitz functions of the form $f_R(x)=f(x)g_R(x)$, $R>0$, where $g_R(x)=h_R(|x|)$ with

$$h_R(t) \ = \begin{cases} 1, & \text{for } \ 0\leq t\leq R,\\ 1-(t-R), & \text{for } \ R\leq t\leq R+1,\\ 0, & \text{for } \ t\geq R.\end{cases}$$

Then $D_r f_R$ is bounded on balls, and in addition $D_r f_R(x)=0$, as long as $|x|>R+r$. Hence, we may apply the previous step and write (6.6) for f_R. Since

$$\begin{aligned}|\nabla f_R(x)| &\leq |f(x)|\,|\nabla g_R(x)| + |g_R(x)|\,|\nabla f(x)|\\ &\leq |f(x)|\,1_{\{R\leq|x|\leq R+1\}} + |\nabla f(x)|,\end{aligned}$$

we get that

$$\iint |f_R(x)-f_R(y)|\,d\mu(x)\,d\mu(y) \ \leq \ \int |\nabla f|\,d\mu_w + \int_{\{R\leq|x|\leq R+1\}} |f|\,d\mu_w. \tag{6.7}$$

By the integrability assumption, the last term is vanishing for growing R. The first term majorizes

$$\int_{\{|x|\le R\}}\int_{\{|y|\le R\}} |f(x)-f(y)|\,d\mu(x)\,d\mu(y).$$

It remains to let $R \to +\infty$, and then (6.7) yields (6.1).

In the general case, consider the functions of the form $f_T = T(f)$, where T is an arbitrary odd, non-decreasing, continuously differentiable function on the real line, such that $T(0) = 0$ and $T' \le 1$. In particular, $|\nabla f_T| \le |\nabla f|$. Note that in terms of the distribution function F of f under μ there is a general identity

$$\frac{1}{2}\iint |T(f(x)) - T(f(y))|\,d\mu(x)\,d\mu(y) = \int_{-\infty}^{+\infty} F(t)(1-F(t))\,\varphi(t)\,dt,$$

where $\varphi = T'$. Hence, by the previous step, being applied to f_T, we obtain that, if f_T is μ_w-integrable,

$$\int_{\mathbb{R}^n} |\nabla f|\,d\mu_w \ \ge\ \int_{-\infty}^{+\infty} F(t)(1-F(t))\,\varphi(t)\,dt \equiv L\varphi.$$

The function φ may be an arbitrary element in the class $\mathcal{F}$ of all even Borel measurable functions on $\mathbb{R}$, such that $0 \le \varphi(t) \le 1$, with an additional assumption that it is continuous and satisfies

$$\int |T(f)|\,d\mu_w = \int_0^{+\infty} \mu_w\{|f| > t\}\,\varphi(t)\,dt < +\infty.$$

Since the function $\psi(t) = \mu_w\{|f| > t\}$ is finite and non-increasing for $t > 0$, a standard density argument allows one to remove the condition $\int_0^{+\infty} \psi(t)\varphi(t)\,dt < +\infty$ when maximizing $L\varphi$. But $\sup_{\varphi\in\mathcal{F}} L\varphi$ is attained for $\varphi \equiv 1$, at which it becomes the right-hand side of (6.6).

c) $\Longleftrightarrow$ *d*): Note that when A and h are fixed, an optimal set in the inequality (6.4) is given by $B = \mathbb{R}^n \setminus A^h$ (which is always closed), and then the inequality becomes

$$\frac{1}{h}\int_{A^h\setminus A} w\,d\mu \ \ge\ \mu(A)\,(1-\mu(A^h)), \tag{6.8}$$

which is exactly (6.3). Indeed, if A and B are non-empty with $h = \mathrm{dist}(A,B) > 0$, then $|x-y| \ge h$, for all $x \in A$, $y \in B$, so $B \subset \mathbb{R}^n\setminus A^h$. From this $A^h\setminus A \subset \mathbb{R}^n\setminus(A\cup B)$, so (6.8) would imply that

$$\frac{1}{h}\int_{\mathbb{R}^n\setminus(A\cup B)} w\,d\mu \ge \frac{1}{h}\int_{A^h\setminus A} w\,d\mu \ \ge\ \mu(A)\,(1-\mu(A^h)) \ \ge\ \mu(A)\mu(B). \tag{6.9}$$

Consequently, *c*) is formally stronger than *d*). Conversely, given a Borel set A and $h > 0$, define $B = \mathbb{R}^n \setminus A^h$. If B is empty, (6.8) is immediate. In the other case, we have that $\mathrm{dist}(A,B) = h$. Indeed, $\mathrm{dist}(A,B) \ge h$, by the definition of B. To derive the converse bound, introduce the function $u(x) = d(A,x)$. We have $u = 0$ on A and $u \ge h$ on B. But the set $J = u(\mathbb{R}^n)$ is connected (as image of the connected space) and therefore represents an interval on the line. Hence, $[0,h] \subset J$. In particular, the level set $u^{-1}(h)$ is non-empty, that is, B contains points x with $d(A,x) = h$. Hence, $\mathrm{dist}(A,B) \le h$. This shows that *c*) is implied by *d*).

c) & *d)* $\iff$ *e)*: In view of the previous step and the general inequalities (6.9), we only need to see that (6.8) may be extended from compact sets to general Borel sets A with finite measure $\mu_w(A)$.

First assume A is a non-empty bounded Borel set in $\mathbb{R}^n$. Then it has a compact closure $\mathrm{clos}(A)$, to which we may apply (6.8):

$$\frac{1}{h}\int_{(\mathrm{clos}(A))^h \setminus \mathrm{clos}(A)} w\,d\mu \,\geq\, \mu(\mathrm{clos}(A))\,(1-\mu((\mathrm{clos}(A))^h).$$

But in general $(\mathrm{clos}(A))^h = A^h$, so the above inequality immediately yields (6.8).

In the general case of a non-empty Borel set A in $\mathbb{R}^n$, one may approximate it by bounded sets $A_R = A \cap \{|x| \leq R\}$, so that by the previous step, for any $h > 0$,

$$\mu_w((A_R)^h) \,\geq\, \mu_w(A_R) + h\,\mu(A_R)\,(1-\mu(A_R)). \tag{6.10}$$

Letting $R \to +\infty$, we have $\mu_w(A_R) \uparrow \mu_w(A)$ and $\mu_w((A_R)^h) \uparrow \mu_w(A^h)$, and similarly for the measure μ. So if $\mu_w(A)$ is finite, the limits in both sides of (6.10) exist, the limit on the right-hand side is finite, and we obtain (6.8) for the set A.

a) $\Rightarrow$ *c)*: Starting from (6.1) and taking into account the previous two steps, we need to derive (6.8) for an arbitrary non-empty compact set A. In this case A^h is bounded and has a finite μ_w-measure for any $h > 0$.

Fix h, and for $\varepsilon > 0$, consider the function

$$f(x) = 1 - \frac{1}{h}\,\min\{d(A^\varepsilon, x), h\}, \quad x \in \mathbb{R}^n.$$

Clearly, $0 \leq f \leq 1$ on the whole space, while $f = 0$ on $B_\varepsilon = \{x \in \mathbb{R}^n : d(A^\varepsilon, x) > h\}$ and $f = 1$ on A^ε. Note that

$$f(x) > t > 0 \Rightarrow d(A^\varepsilon, x) < h \Rightarrow x \in A^{h+\varepsilon},$$

so $\mu_w\{f > t\} \leq \mu_w(A^{h+\varepsilon}) < +\infty$, that is, f is μ_w-finite. Therefore, we may apply (6.1) to this function. By the construction, $|\nabla f| \leq 1/h$ on the whole space, but since both A^ε and B_ε are open and f is constant on these sets, $|\nabla f| = 0$ on $A^\varepsilon \cup B_\varepsilon$. Hence (6.1) gives

$$\frac{1}{h}\int_{\mathbb{R}^n \setminus (A^\varepsilon \cup B_\varepsilon)} w\,d\mu \,\geq\, \frac{1}{2}\iint |f(x) - f(y)|\,d\mu(x)\,d\mu(y) \,\geq\, \mu(A^\varepsilon)\mu(B_\varepsilon). \tag{6.11}$$

By the closeness of A, we have $\mu(A^\varepsilon) \downarrow \mu(A)$ and $B^\varepsilon \uparrow \mathbb{R}^n \setminus \mathrm{clos}(A^h)$, as $\varepsilon \to 0$. Indeed, the complement $\bar{B}_\varepsilon = \mathbb{R}^n \setminus B^\varepsilon = \{x : d(A^\varepsilon, x) \leq h\}$ contains A^h and therefore $\mathrm{clos}(A^h)$. Conversely, if $d(A^\varepsilon, x) \leq h$, then $x \in A^{h'+\varepsilon}$, for any $h' > h$. In particular, $x \in A^{h+2\varepsilon}$. Hence, $\bar{B}_\varepsilon \subset A^{h+2\varepsilon}$ and

$$\bigcap_{\varepsilon>0} \bar{B}_\varepsilon \subset \bigcap_{\varepsilon>0} A^{h+2\varepsilon} = \bigcap_{\varepsilon>0} (A^h)^{2\varepsilon} = \mathrm{clos}(A^h).$$

The two inclusions give $\bigcap_{\varepsilon>0} \bar{B}_\varepsilon = \mathrm{clos}(A^h)$. In particular, $\mu(B^\varepsilon) \uparrow 1 - \mu(\mathrm{clos}(A^h))$. Taking the limit in (6.11), we get

$$\frac{1}{h}\int_{\mathrm{clos}(A^h)\setminus A} w\,d\mu \,\geq\, \mu(A)\,(1-\mu(\mathrm{clos}(A^h))).$$

Now apply this bound with arbitrary $h' \in (0, h)$. Using $\mathrm{clos}(A^{h'}) \subset A^h$, we obtain a similar inequality

$$\frac{1}{h'} \int_{A^h \setminus A} w \, d\mu \, \geq \, \mu(A) \, (1 - \mu(A^h)). \tag{6.12}$$

It remains to let $h' \to h$, and then we arrive at the desired inequality (6.8).

c) $\Rightarrow$ *b*): Let A be closed with finite measure $\mu_w(A)$. Since, as explained before, the property *c*) may equivalently be written in the form (6.8), we get that, for any $\varepsilon > 0$,

$$\frac{\mu_w(A^\varepsilon) - \mu_w(A)}{\varepsilon} \, \geq \, \mu(A) \, (1 - \mu(A^\varepsilon)).$$

It remains to send ε to zero and note that $\mu(A^\varepsilon) \downarrow \mu(A)$.

Since *b*) is stronger than *b'*), we have covered the whole cycle: *a*) $\Leftrightarrow$ *b'*) $\Rightarrow$ *c*), *d*), *e*) $\Rightarrow$ *b*) $\Rightarrow$ *b'*). Proposition 6.1 is proved. □

7. Localization

In order to extend Corollary 5.2 to higher dimensions, we use a localization argument in the form of R. Kannan, L. Lovász and M. Simonovits, described in [33] and [27] (cf. also [22, 23] for further developments). The method itself allows one to reduce various multidimensional functional and geometric inequalities to specific problems in dimension one. In particular, we have:

Proposition 7.1. *Let $\alpha, \beta > 0$ and $\kappa \in [-\infty, 1]$ be fixed, and let f_1, f_2, f_3, f_4 be non-negative continuous functions on $\mathbb{R}^n$. Then the inequality of the form*

$$\Big(\int f_1 \, d\mu \Big)^\alpha \Big(\int f_2 \, d\mu \Big)^\beta \leq \Big(\int f_3 \, d\mu \Big)^\alpha \Big(\int f_4 \, d\mu \Big)^\beta \tag{7.1}$$

holds true for all κ-concave probability measures μ on $\mathbb{R}^n$, if it holds true in the class of all one-dimensional κ-concave probability measures on $\mathbb{R}^n$ with a compact support.

In the log-concave case (when $\kappa = 0$), this remarkable observation was made in [27]. But the general case is no more difficult, so we omit the proof. Let us just mention that the argument is based on the so-called localization lemma of Lovász-Simonovits [33].

In many interesting situations, the continuity assumption on the functions f_i's in Proposition 7.1 may be relaxed (by using suitable approximations). This is so in the following particular case.

Let $\alpha, \beta > 0$, $h > 0$, and $\kappa \in [-\infty, 1]$ be fixed, and let v, w be non-negative continuous functions on $\mathbb{R}^n$.

Corollary 7.2. *For all non-empty Borel sets $A, B \subset \mathbb{R}^n$ at distance h, and for any κ-concave probability measure μ on $\mathbb{R}^n$, we have that*

$$\mu(A)^\alpha \, \mu(B)^\beta \, \leq \, \Big(\int_{\mathbb{R}^n \setminus (A \cup B)} w \, d\mu \Big)^\alpha \Big(\int_{\mathbb{R}^n} v \, d\mu \Big)^\beta \tag{7.2}$$

if and only if this property holds true in the class of all one-dimensional κ-concave probability measures on $\mathbb{R}^n$ with a compact support.

Proof. Assume the property (7.2) is fulfilled in the class of all one-dimensional κ-concave probability measures on $\mathbb{R}^n$, and let μ_0 be an arbitrary κ-concave probability measure on $\mathbb{R}^n$ with compact support. Once (7.2) is established for compactly supported measures, the general case would follow immediately by applying (7.2) to the normalized restrictions of κ-concave measures to the balls of large radii.

Using regularity of Borel measures, it is enough to establish (7.2) for μ_0, when A and B are non-empty compact sets in $\mathbb{R}^n$ at distance $\mathrm{dist}(A,B) > h$.

Choose $\varepsilon > 0$ small enough, such that the open (Euclidean) neighbourhoods A^ε and B^ε of these sets are still at a distance more than h. Take two continuous functions $f_1, f_2 : \mathbb{R}^n \to [0,1]$ such that $f_1 = 1$ on A, $f_1 = 0$ on $\mathbb{R}^n \setminus A^\varepsilon$, and similarly $f_2 = 1$ on B, $f_2 = 0$ on $\mathbb{R}^n \setminus B^\varepsilon$, and take an arbitrary continuous function $\varphi : \mathbb{R}^n \to [0,1]$, such that $\varphi = 1$ on the closed set $C_\varepsilon = \mathbb{R}^n \setminus (A^\varepsilon \cup B^\varepsilon)$. Then

$$\mu(A) \leq \int f_1\, d\mu \leq \mu(A^\varepsilon), \qquad \mu(B) \leq \int f_2\, d\mu \leq \mu(B^\varepsilon). \tag{7.3}$$

By the one-dimensional hypothesis (7.2), applied to A^ε and B^ε, and using the right inequalities in (7.3), we obtain that

$$\Big(\int f_1\, d\mu\Big)^\alpha \Big(\int f_2\, d\mu\Big)^\alpha \ \leq \ \Big(\int w\,\varphi\, d\mu\Big)^\alpha \Big(\int v\, d\mu\Big)^\beta, \tag{7.4}$$

which holds for any one-dimensional κ-concave probability measure μ on $\mathbb{R}^n$ with a compact support. Hence, by Proposition 7.1, (7.4) holds for μ_0, as well. Using the left inequalities in (7.3), we therefore obtain that

$$\mu_0(A)^\alpha\, \mu_0(B)^\beta \ \leq \ \Big(\int w\,\varphi\, d\mu_0\Big)^\alpha \Big(\int v\, d\mu_0\Big)^\beta.$$

Finally, taking here the infimum over all admissible φ's and using closeness of the set C_ε, we arrive at

$$\begin{aligned}\mu_0(A)^\alpha\, \mu_0(B)^\beta &\leq \Big(\int_{C_\varepsilon} w\, d\mu_0\Big)^\alpha \Big(\int v\, d\mu_0\Big)^\beta \\ &\leq \Big(\int_{\mathbb{R}^n \setminus (A\cup B)} w\, d\mu_0\Big)^\alpha \Big(\int v\, d\mu_0\Big)^\beta,\end{aligned}$$

which is the statement. Corollary 7.2 follows. □

8. Proof of Theorems 1.1-1.2

We are prepared to consider weighted analytic inequalties for κ-concave measures on $\mathbb{R}^n$. Let X be a random vector with distribution μ. Define

$$\|X\|_0 = \exp \mathbb{E}\, \log |X|.$$

In view of the general Proposition 6.1 (and recalling Remark 6.2), Theorem 1.1 may equivalently be formulated as the following geometric statement which we already mentioned in Section 1, cf. (1.5).

Theorem 8.1. *Let μ be a κ-concave probability measure on $\mathbb{R}^n$, $-\infty < \kappa \leq 0$. For all non-empty Borel sets $A, B \subset \mathbb{R}^n$ at distance $h > 0$,*

$$\mu(A)\,\mu(B) \ \leq \ \frac{C_\kappa}{h} \int_{\mathbb{R}^n \setminus (A\cup B)} (\|X\|_0 - \kappa |x|)\, d\mu(x), \tag{8.1}$$

where the constants C_κ continuously depend on κ in the range $\kappa \leq 0$.

Proof. First let μ be a κ-concave probability measure on $\mathbb{R}$ with a compact support. By Corollary 5.2 in the form of the inequality (5.6), for any locally Lipschitz function f on the real line, we readily obtain that

$$\frac{1}{2}\iint |f(x)-f(y)|\,d\mu(x)d\mu(y)\ \le (1+C_\kappa\,\|X\|_0)\int |f'(x)|\,(1-\kappa|x|)\,d\mu(x)$$

with some continuous function $C = C_\kappa$, $\kappa \le 0$. We are in a position to apply Proposition 6.1 in dimension one, which gives

$$\mu(A)\,\mu(B)\ \le\ \frac{1+C_\kappa\,\|X\|_0}{h}\int_{\mathbb{R}\setminus(A\cup B)}(1-\kappa|x|)\,d\mu(x) \tag{8.2}$$

for any $h > 0$ and for all non-empty Borel sets $A, B \subset \mathbb{R}$ at distance h. Note that the condition $\int_A (1-\kappa|x|)\,d\mu(x) < +\infty$ in the claim d) of Proposition 6.1 is satisfied.

In order to extend (8.2) to higher dimensions, we have to modify this inequality. Note that, for any random variable $\xi \ge 0$ with finite L^q-norm, $q > 0$,

$$1+\|\xi\|_0 \le \|1+\xi\|_0 \le \|1+\xi\|_q.$$

The first bound can be obtained from Jensen's inequality when applying it to the convex function $t \to \log(1+e^t)$. Taking $\xi = 1 + C_\kappa\,|X|$, (8.2) yields

$$\mu(A)\,\mu(B)\ \le\ \frac{\|1+C_\kappa\,|X|\,\|_q}{h}\int_{\mathbb{R}\setminus(A\cup B)}(1-\kappa|x|)\,d\mu(x).$$

Assume $0 < q \le 1$. Using $\mu(B) \ge (\mu(B))^{1/q}$ and raising the above inequality to the power q, we get

$$(\mu(A))^q\,\mu(B)\ \le\ \int\Big(\frac{1+C_\kappa\,|x|}{h}\Big)^q\,d\mu(x)\,\Big(\int_{\mathbb{R}\setminus(A\cup B)}(1-\kappa|x|)\,d\mu(x)\Big)^q. \tag{8.3}$$

Now, we are in a position to apply Corollary 7.2 with $\alpha = q$, $\beta = 1$, $w(x) = 1-\kappa|x|$, and $v(x) = (\frac{1+C_\kappa\,|x|}{h})^q$, to get that

$$(\mu(A))^q\,\mu(B)\ \le\ \frac{\|1+C_\kappa\,|X|\,\|_q^q}{h^q}\Big(\int_{\mathbb{R}^n\setminus(A\cup B)}(1-\kappa|x|)\,d\mu(x)\Big)^q \tag{8.4}$$

for any κ-concave probability measure μ on $\mathbb{R}^n$ and for all non-empty Borel sets $A, B \subset \mathbb{R}^n$ at distance $h > 0$ (where X is a random vector in $\mathbb{R}^n$ with distribution μ).

Indeed, according to Corollary 7.2 it suffices to establish (8.4) for one-dimensional κ-concave μ with a compact support. Such measures may be characterized as the distributions of $X = a + Y\theta$ with arbitrary orthogonal vectors $a, \theta \in \mathbb{R}^n$, $|\theta| = 1$, and where Y is a random variable with an arbitrary compactly supported κ-concave distribution on $\mathbb{R}$. Note that the Euclidean norm $|X|$ of X in $\mathbb{R}^n$ satisfies

$$|X|^2 = |a|^2 + |Y|^2 \ge |Y|^2,$$

so $\|1+|X|\,\|_q \ge \|1+|Y|\,\|_q$. Therefore, (8.4) for the one-dimensional random vector X is implied by the one-dimensional inequality (8.3) for the distribution μ of the random variable Y.

Thus, we have obtained the inequality (8.4) for the class of all κ-concave probability measures μ on $\mathbb{R}^n$. To proceed, we need to distinguish between small and large values of $-\kappa$.

Case 1. $-\frac{1}{2} \le \kappa \le 0$.

In this case one may just take $q = 1$. By Corollary 4.3, cf. (4.8), applied with the Euclidean norm in $\mathbb{R}^n$, and using continuity of C_κ in κ, we then have

$$\|1 + C_\kappa\,|X|\,\|_1 = 1 + C_\kappa\,\|X\|_1 \le 1 + C\|X\|_0$$

with some absolute constant C. Hence, (8.4) yields

$$\mu(A)\,\mu(B) \ \le\ \frac{1 + C\|X\|_0}{h} \int_{\mathbb{R}^n\setminus(A\cup B)} (1 - \kappa|x|)\,d\mu(x). \tag{8.5}$$

Case 2. $\kappa \le -\frac{1}{2}$.

In this case we take $q = -1/(2\kappa)$. Since in general $\|1 + \xi\|_q^q \le 1 + \|\xi\|_q^q$ (where $\xi \ge 0$ and $0 < q \le 1$), again applying Corollary 4.3, i.e., $\|X\|_q \le C(\kappa,q)\,\|X\|_0$, we obtain that

$$\|1 + C_\kappa\,|X|\,\|_q^q \ \le\ 1 + C_\kappa^q\,\|X\|_q^q \ \le\ 1 + C_\kappa^q\,C(\kappa,q)^q\,\|X\|_0^q.$$

Recall that the constants $C(\kappa,q)$ are finite and continuously varying in the region $q < -1/\kappa$, $\kappa \le 0$. Hence with $q = -1/(2\kappa)$, (8.4) yields

$$(\mu(A))^q\,\mu(B) \ \le\ \frac{1 + C_\kappa^q\,\|X\|_0^q}{h^q} \bigg(\int_{\mathbb{R}^n\setminus(A\cup B)} (1 - \kappa|x|)\,d\mu(x)\bigg)^q$$

with some (new) continuous function C_κ in the region $\kappa \le -\frac{1}{2}$. Raising this inequality to the power $1/q$ and using $(a^q + b^q)^{\frac{1}{q}} \le 2^{\frac{1}{q}-1}(a+b)$, we get that

$$\mu(A)\,(\mu(B))^{1/q} \ \le\ 2^{\frac{1}{q}-1}\,\frac{1 + C_\kappa\,\|X\|_0}{h} \int_{\mathbb{R}^n\setminus(A\cup B)} (1 - \kappa|x|)\,d\mu(x). \tag{8.6}$$

Note that one may interchange A and B in (8.6) so that to bound its right-hand side by $\mu(B)\,(\mu(A))^{1/q}$. So, one may assume without loss in generality that $\mu(B) \ge 1/2$. But then $\mu(A)\,(\mu(B))^{1/q} \ge 2^{1-\frac{1}{q}}\,\mu(A)\,\mu(B)$, and (8.6) gives

$$\mu(A)\,\mu(B) \ \le\ 4^{-2\kappa-1}\,\frac{1 + C_\kappa\,\|X\|_0}{h} \int_{\mathbb{R}^n\setminus(A\cup B)} (1 - \kappa|x|)\,d\mu(x). \tag{8.7}$$

This inequality is very similar to (8.5), so both can further be treated in a similar manner. It remains to make them to be homogeneous with respect to X. If we apply (8.7) to the random vector λX, $\lambda > 0$, and to the sets λA, λB, the inequality will take the form

$$\mu(A)\,\mu(B) \ \le\ 4^{-2\kappa-1}\,\frac{1 + C_\kappa\,\lambda\|X\|_0}{\lambda h} \int_{\mathbb{R}^n\setminus(A\cup B)} (1 - \lambda\kappa|x|)\,d\mu(x).$$

Then choose $\lambda = 1/\|X\|_0$ to rewrite it as

$$\mu(A)\,\mu(B) \ \le\ 4^{-2\kappa-1}\,\frac{1 + C_\kappa}{h} \int_{\mathbb{R}^n\setminus(A\cup B)} (\|X\|_0 - \kappa|x|)\,d\mu(x).$$

It is exactly of the desired form (8.1). The case of small κ may be included in the above inequality with $C_\kappa = C$ (and without term $4^{-2\kappa-1}$).

Theorem 8.1 is proved. $\square$

Remark. In [12] it is shown that, if μ is a κ-concave probability measure on $\mathbb{R}^n$, $-\infty < \kappa < 0$, then for any locally Lipschitz function f on $\mathbb{R}^n$,

$$\iint |f(x) - f(y)|\,d\mu(x)d\mu(y) \ \le\ C\cdot(1-\kappa) \int \big|\nabla f(x)\big|\big(r + |x|\big)\,d\mu(x), \tag{8.8}$$

where C is a universal constant and r is a quantile of the Euclidean norm under μ of order $2/3$ (which is larger than, but still equivalent to the median). As we know from Corollary 4.2, r may be replaced with the geometric mean $m_0 = \|X\|_0$. Therefore, up to constants, depending on κ, the inequalities (8.8) and (1.4) of Theorem 1.1 are equivalent in the case where the parameter κ is separated from zero.

Note that (8.8) implies the weighted Poincaré-type inequality

$$\mathrm{Var}_\mu(f) \,\le\, C \cdot (1-\kappa)^2 \int \big|\nabla f(x)\big|^2 \big(r^2 + |x|^2\big)\, d\mu(x)$$

with some (other) universal constant C. This inequality is also derived in [12], where however the factor $1-\kappa = \frac{\beta-n+1}{\beta-n}$ missed the power 2 in the formulation of Theorem 5.1.

Proof of Theorem 1.2. Note that in general

$$\iint |f(x) - f(y)|\, d\mu(x)\, d\mu(y) \ge \int |f - m(f)|\, d\mu,$$

where $m(f)$ is a median of f under μ. Hence, the inequality (1.4) of Theorem 1.1 yields

$$\int |f|\, d\mu \le \int |\nabla f|\, w\, d\mu \tag{8.9}$$

for any locally Lipschitz function f on $\mathbb{R}^n$ with μ-median zero, where we use $w(x) = C_\kappa(m_0 - \kappa|x|)$ to denote the corresponding weight function (with $m_0 = \|X\|_0$). In particular, if additionally $f \ge 0$, we may apply (8.9) to f^2 and get, together with Cauchy's inequality, that

$$\int f^2\, d\mu \,\le\, 2\int f\, |\nabla f|\, w\, d\mu \,\le\, 2\Big(\int f^2\, d\mu\Big)^{1/2} \Big(\int |\nabla f|^2\, w^2\, d\mu\Big)^{1/2}.$$

Hence, $\int f^2\, d\mu \le 4 \int |\nabla f|^2\, w^2\, d\mu$. In the general case of a locally Lipschitz function f with median zero, represent $f = f^+ = f^-$ with $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$, and apply the obtained bound to f^+ and f^-. This gives

$$\int_{\{f \ge 0\}} f^2\, d\mu \le 4 \int_{\{f \ge 0\}} |\nabla f|^2\, w^2\, d\mu,$$
$$\int_{\{f \le 0\}} f^2\, d\mu \le 4 \int_{\{f \le 0\}} |\nabla f|^2\, w^2\, d\mu.$$

There is no loss in generality to assume that $|\nabla f| = 0$ on the set $\{f = 0\}$, since otherwise one may consider approximations of f by the functions of the form $T(f)$ with $T(0) = T'(0) = 0$. Consequently, adding the above inequalities, we arrive at

$$\mathrm{Var}_\mu(f) \,\le\, \int f^2\, d\mu \,\le\, 4\int |\nabla f|^2\, w^2\, d\mu.$$

It remains to note that $w^2 \le 2C_\kappa^2\, (m_0^2 + \kappa^2 |x|^2)$. □

Appendix

Here we give details, justifying two steps in the proof of Proposition 6.1. One of the steps was based on the well-known assertion that, in order to derive an isoperimetric inequality from a given analytic inequality, one can properly approximate indicator functions by Lipschitz functions. For our purposes, we need Lemma 9.1 below, which extends Lemma 3.5 of [11] to the case of infinite measures. The second step made use of the so-called coarea inequality, Lemma 9.2 below, which extends Lemma 3.2 of [11]. The coarea equality for the class of (sufficiently) smooth Lipschitz functions is a classical tool in the Theory of Sobolev spaces, cf. e.g. [34]. However, it becomes an inequality when stated for more general classes of functions and metric spaces.

Thus, assume we have a metric space (M,d), equipped with a σ-finite Borel measure ν. To any function f on M, one may associate its generalized modulus of the gradient defined to be

$$|\nabla f(x)| = \limsup_{y \to x} \frac{|f(x) - f(y)|}{|x - y|}$$

for points $x \in M$ that are not isolated, and to be zero otherwise. Note the collection of all isolated points forms an open subset of M.

As easy to see, if f is continuous on M, then $|\nabla f|$ is Borel measurable. We say that f is locally Lipschitz, if any point in M has a neighbourhood, where f has a finite Lipschitz seminorm. In this case $|\nabla f|$ is everywhere finite. Moreover, when M is locally compact, the generalized modulus of the gradient of any locally Lipschitz function is bounded on every ball in M.

For every Borel set A in M we define its ν-perimeter (or, the outer Minkowski content) as

$$\nu^+(A) \, = \, \liminf_{\varepsilon \downarrow 0} \, \frac{\nu(A^\varepsilon \setminus A)}{\varepsilon},$$

where $A^\varepsilon = \{x \in \mathbb{R}^n : \exists y \in A, \, d(x,y) < \varepsilon\}$ denotes an open Euclidean ε-neighbourhood of A. In particular, $\nu^+(\emptyset) = \nu^+(M) = 0$.

Lemma 9.1. *For any Borel set A in M, there exists a sequence of Lipschitz functions $f_n : M \to [0,1]$, such that $f_n \to 1_{\mathrm{clos}(A)}$ pointwise and*

$$\limsup_{n \to \infty} \int |\nabla f_n| \, d\nu \leq \nu^+(A).$$

Moreover, if $\nu(A^\varepsilon) < +\infty$, for some $\varepsilon > 0$, such functions may be chosen to satisfy the property that $\nu\{|f_n| > t\} < +\infty$ for all $t > 0$.

Proof. In general, $\mathrm{clos}(A^\varepsilon) \subset A^{\varepsilon'}$ whenever $0 < \varepsilon < \varepsilon'$, where $\mathrm{clos}(A)$ denotes the closure of A. Therefore, by the definition of the perimeter, one can always choose a sequence $\varepsilon_n \downarrow 0$ such that

$$\lim_{n \to \infty} \frac{\nu(\mathrm{clos}(A^{\varepsilon_n}) \setminus A)}{\varepsilon_n} = \nu^+(A).$$

Now, take a sequence $c_n \in (0,1)$, such that $c_n \to 0$, and define the functions

$$f_n(x) = \max\bigg\{1 - \frac{d(A^{c_n \varepsilon_n}, x)}{(1 - c_n)\varepsilon_n}, 0\bigg\},$$

where $d(B,x) = \inf_{y \in B} d(y,x)$. All distance functions have Lipschitz seminorm ≤ 1, so $\|f_n\|_{\mathrm{Lip}} \leq 1/((1-c_n)\varepsilon_n)$ and

$$|\nabla f_n(x)| \leq \frac{1}{(1 - c_n)\varepsilon_n}, \quad \text{for all } x \in M.$$

Moreover, $|\nabla f_n| = 0$ on A, since $f_n = 1$ on the open set $A^{c_n\varepsilon_n}$ containing A. If $x \notin A^{\varepsilon_n}$, then $d(A,x) \geq \varepsilon_n$, and by the triangle inequality, $d(A^{c_n\varepsilon_n}, x) \geq (1-c_n)\varepsilon_n$. This shows that $f_n = 0$ outside A^{ε_n}, so $|\nabla f_n| = 0$ on the open set $M \setminus \mathrm{clos}(A^{\varepsilon_n})$.

Combining these properties, we conclude that

$$\int_M |\nabla f_n|\, d\nu = \int_{\mathrm{clos}(A^{\varepsilon_n})\setminus A} |\nabla f_n|\, d\nu \leq \frac{\nu(\mathrm{clos}(A^{\varepsilon_n}) \setminus A)}{(1-c_n)\varepsilon_n}.$$

Taking the limit, we arrive at the desired inequality.

Finally, assume $\nu(A^\varepsilon) < +\infty$ for some $\varepsilon > 0$. Hence, $\nu(A^{\varepsilon_n}) < +\infty$ for all n large enough. For the constructed sequence f_n, the required property $\nu\{|f_n| > t\} < +\infty$, where we may assume $0 < t \leq 1$, reads as $\nu\{d(A^{c_n\varepsilon_n}, x) < (1-t)(1-c_n)\varepsilon_n\} < +\infty$. It is fulfilled, since by the triangle inequality, the property $d(A^{c_n\varepsilon_n}, x) < (1-t)(1-c_n)\varepsilon_n$ implies $d(A,x) < t(1-c_n)\varepsilon_n < \varepsilon_n$, so $x \in A^\varepsilon$.

Lemma 9.1 is proved. □

Lemma 9.2. *Given a continuous function f on M, let $\nu\{|f| > t\} < +\infty$ for all $t > 0$, and assume the function*

$$D_r f(x) = \sup_{0<|x-y|<r} \frac{|f(x) - f(y)|}{|x-y|}, \qquad x \in M, \tag{9.1}$$

is ν-integrable for some $r > 0$. Then

$$\int |\nabla f|\, d\nu \geq \int_{-\infty}^0 \nu^+\{f < t\}\, dt + \int_0^{+\infty} \nu^+\{f > t\}\, dt. \tag{9.2}$$

Proof. Consider the case $f \geq 0$. First assume f is ν-integrable and bounded. If we subtract from f the constant $a = \inf f$, the function $g = f - a$ will satisfy the same hypothesis as f, while (9.2) will not change. Hence we may also assume $\inf f = 0$ (which is necessary, when the measure ν is not finite).

Put $b = \sup f$ and define a non-increasing family of open sets $A_t = \{f > t\}$ in M with parameter $t \in (0,b)$. Since

$$\int f\, d\nu = \int_0^b \nu(A_t)\, dt < +\infty,$$

necessarily $\nu(A_t) < +\infty$ for all $t \in (0,b)$. In this case we may write (using rational numbers $h > 0$)

$$\nu^+(A_t) = \sup_{\varepsilon>0} \inf_{0<h<\varepsilon} \frac{\nu(A_t^h) - \nu(A_t)}{h} = \liminf_{h\downarrow 0} \frac{\nu(A_t^h) - \nu(A_t)}{h}.$$

This representation shows (since $\nu(A_t)$ and $\nu(A_t^h)$ are monotone with respect to t) that the resulting function $t \to \nu^+(A_t)$ is Borel measurable on $(0,b)$ and thus on the whole real line.

Introduce the family of lower semicontinuous functions $f_\varepsilon(x) = \sup_{|x-y|<\varepsilon} f(y)$, $\varepsilon > 0$, so that

$$\limsup_{\varepsilon\to 0} \frac{f_\varepsilon(x) - f(x)}{\varepsilon} \leq |\nabla f(x)|, \qquad x \in M.$$

By the definition (9.1),

$$\sup_{0<\varepsilon<r} \frac{f_\varepsilon(x) - f(x)}{\varepsilon} = \sup_{0<|x-y|<r} \frac{f(y) - f(x)}{|x-y|} \leq D_r f(x),$$

so $(f_\varepsilon - f)/\varepsilon$ for $\varepsilon < r$ have a common ν-integrable majorant, if r is small enough. Hence, we may apply the Lebesgue dominated convergence theorem, which gives

$$\limsup_{\varepsilon\to 0} \int \frac{f_\varepsilon - f}{\varepsilon}\, d\nu \,\leq\, \int |\nabla f(x)|\, d\nu.$$

Now, apply the identity $\{f_\varepsilon > t\} = \{f > t\}^\varepsilon$ and use once more ν-integrability of f to write

$$\int \frac{f_\varepsilon - f}{\varepsilon}\, d\nu = \int_0^b \frac{\nu(A_t^\varepsilon) - \nu(A_t)}{\varepsilon}\, dt.$$

By Fatou's lemma,

$$\liminf_{\varepsilon\to 0} \int_0^b \frac{\nu(A_t^\varepsilon) - \nu(A_t)}{\varepsilon}\, dt \geq \int_0^b \liminf_{\varepsilon\to 0} \frac{\nu(A_t^\varepsilon) - \nu(A_t)}{\varepsilon}\, dt = \int_0^b \nu^+(A_t)\, dt.$$

The two inequalities yield (9.2).

To remove the condition of the ν-integrability of f, just assume f is non-negative and bounded as before, with $\inf f = 0$, $\sup f = b$. Consider functions of the form $f_T = T(f)$ where T is an arbitrary non-decreasing, continuously differentiable function on $[0, b]$, such that $T(0) = 0$ and $T' \leq 1$. Then $|\nabla f_T| \leq |\nabla f|$ and $D_r f_T \leq D_r f$, so $D_r f_T$ is ν-integrable. By the previous step, applied to f_T, we obtain that, if f_T is ν-integrable,

$$\int |\nabla f|\, d\nu \,\geq\, L\varphi \equiv \int_a^b \nu^+\{f > t\}\, \varphi(t)\, dt,$$

where $\varphi = T'$. Note that the function φ may be an arbitrary element in the class $\mathcal{F}$ of all Borel measurable functions on $[0, b]$, such that $0 \leq \varphi(t) \leq 1$, with an additional assumption that it is continuous and satisfies

$$\int T(f)\, d\nu = \int_0^b \nu\{f > t\}\, \varphi(t)\, dt < +\infty.$$

Since the function $\psi(t) = \nu\{f > t\}$ is finite and non-increasing on $(0, b)$, one may apply a simple density argument to remove the condition $\int_0^b \psi(t)\varphi(t)\, dt < +\infty$ when maximizing $L\varphi$. But $\sup_{\varphi\in\mathcal{F}} L\varphi$ is exactly the second integral on the right-hand side of (9.2).

Now, to remove the condition of the boundedness of f, in the general situation one may truncate f at a high level $c > 0$. That is, apply (9.2) to the new function f_c defined to be $f_c(x) = f(x)$, if $f(x) \leq c$, and $f_c(x) = c$, if $f(x) > c$. Then $|\nabla f_c| = 0$ on the open set $\{f > c\}$, and in addition $|\nabla f_c| \leq |\nabla f|$, $D_r f_c \leq D_r f$ on the whole space. In particular, $D_r f_c$ is ν-integrable. Choosing c so that $\nu\{f = c\} = 0$ (this is the only place, where we need σ-finiteness of the measure), (9.2) yields

$$\int_{\{f(x)\leq c\}} |\nabla f(x)|\, d\nu(x) \,\geq\, \int_0^c \nu^+\{x \in M : f(x) > t\}\, dt.$$

Letting $c \to +\infty$ along admissible values, we arrive at

$$\int |\nabla f|\, d\nu \,\geq\, \int_0^{+\infty} \nu^+\{f > t\}\, dt, \tag{9.3}$$

which is (9.2) for f. Note that one may formally sharpen this inequality by writing

$$\int_{\{f>0\}} |\nabla f|\, d\nu \geq \int_0^{+\infty} \nu^+\{f > t\}\, dt. \tag{9.4}$$

Indeed, apply (9.3) to $f_n = T_n(f)$, where $T_n : [0,+\infty) \to [0,+\infty)$ are increasing smooth functions, such that $T_n(0) = T_n'(0) = 0$, $T_n' \le 1$. Then $|\nabla f_n| \le |\nabla f|$ everywhere, $|\nabla f_n| = 0$, as long as $f = 0$, and we get

$$\int_{\{f>0\}} |\nabla f|\, d\nu \ \ge \int_0^{+\infty} \nu^+\{f > t\}\, \varphi_n(t)\, dt, \tag{9.5}$$

where $\varphi_n = T_n'$. Choosing these functions so that, for all $t > 0$, $\varphi_n(t) \uparrow 1$ as $n \to \infty$, in the limit (9.5) will become the desired inequality (9.4).

Finally, consider the general case and write $f = f^+ - f^-$, where $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$. Clearly, $|\nabla f^\pm| \le |\nabla f|$ and $D_r f^\pm \le D_r f$, so we may apply to the functions $f^\pm$ the previous step in the form of (9.4) to get that

$$\int_{\{f>0\}} |\nabla f|\, d\nu \ge \int_0^{+\infty} \nu^+\{f > t\}\, dt,$$

$$\int_{\{f<0\}} |\nabla f|\, d\nu \ge \int_{-\infty}^{0} \nu^+\{f < t\}\, dt.$$

It remains to add these two inequalities. Lemma 9.2 is proved. □

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