Journal of Mathematical Sciences, Vol. 159, No. 1, 2009

ON THE ISOPERIMETRIC CONSTANTS FOR PRODUCT MEASURES

Sergey G. Bobkov
University of Minnesota
127 Vincent Hall, Minneapolis, MN, 55455, USA
bobko001@umn.edu

UDC 517.9

Dedicated to Nina Nikolaevna Uraltseva

Isoperimetric constants of product probability measures are known to have an almost dimension-free character. We propose a new proof based on certain Sobolev-type inequalities of additive type (introduced by the author in connection with the isoperimetric problem in Gauss space). Bibliography: 18 *titles.*

Let (M, ρ, μ) be a separable metric space equipped with a normalized (probability) Borel measure, i.e., $\mu(M) = 1$. For every Borel subset A of M and real $\varepsilon > 0$ we define its μ-perimeter

$$\mu^+(A) = \liminf_{\varepsilon \to 0} \frac{\mu(A^\varepsilon) - \mu(A)}{\varepsilon},$$

where

$$A^\varepsilon = \{x \in M : \rho(x, a) < \varepsilon, \text{ for some } a \in A\}$$

stands for an open ε-neighborhood of A. The isoperimetric constant of the triple (M, ρ, μ) represents an optimal value $h = h_\mu$ in the isoperimetric type inequality

$$\mu^+(A) \geqslant h \min\{\mu(A), 1 - \mu(A)\}, \qquad A \subset M \text{ Borel}. \tag{1}$$

This quantity was introduced in 1969 by Cheeger [**1**] to bound from below the spectral gap of the Laplacian on compact Riemannian manifolds, and nowadays (1) is often called an isoperimetric inequality of the Cheeger type. The relationship between more general isoperimetric and certain Sobolev type inequalities was earlier considered by Maz'ya [**2**] (see for history, for example, [**3, 4**]). What was noticed in [**1**] as an equivalent functional form for (1) (in the framework of Riemannian manifolds) is that, for any smooth function f on M with μ-median m, we have

$$h \int |f - m| \, d\mu \leqslant \int |\nabla f| \, d\mu. \tag{2}$$

The isoperimetric constants play also an important role in other Sobolev type inequalities. Therefore, it is natural to ask how these quantities reflect the dimension of the space and in particular how they behave on product spaces.

Let $M^n = M \times \cdots \times M$ denote the nth Cartesian power of M equipped with some metric ρ_n generating the canonical product topology and with the product measure $\mu^n = \mu \otimes \cdots \otimes \mu$.

Translated from *Problemy Matematicheskogo Analiza*, **40**, May 2009, pp. 49–56.

For any function f on M (and similarly on M^n with respect to ρ_n), we define the "modulus of the gradient" to be

$$|\nabla f(x)| = \limsup_{\rho(x,y)\to 0} \frac{|f(x)-f(y)|}{\rho(x,y)}, \qquad x \in M,$$

with the convention that $|\nabla f(x)| = 0$ as long as the point x is isolated in M. Note that $|\nabla f|$ is a Borel measurable, finite function, whenever f is locally Lipschitz. By saying "locally Lipschitz" we mean that f has a finite Lipschitz constant on every ball in M.

We assume the metric ρ_n is consistent with the measure μ^n in the sense that, for any Lipschitz function f on M^n, μ^n-almost everywhere

$$|\nabla f(x)|^2 = \sum_{i=1}^n |\nabla_{x_i} f(x)|^2. \tag{3}$$

Here, $|\nabla_{x_i} f|$ denotes the modulus of the gradient for the function $x_i \to f(x)$ on M with fixed x_j, $j \neq i$. In reasonable situations the metric ρ_n may be chosen to be of the Euclidean type:

$$\rho_n^2(x,y) = \sum_{i=1}^n \rho^2(x_i,y_i), \qquad x = (x_1,\dots,x_n),\ y = (y_1,\dots,y_n) \in M^n.$$

For example, this is the case where $M = \mathbf{R}^d$ with usual Euclidean distance and with an arbitrary probability measure, which is absolutely continuous with respect to Lebesgue measure on $\mathbf{R}^d$. The particular case of the line $M = \mathbf{R}$ is of a special interest.

With the above general assumptions and notations, we have the following:

Theorem 1. *Under the hypothesis* (3), *for any integer* $n \geqslant 1$,

$$h_{\mu^n} \geqslant \frac{1}{2\sqrt{6}}\, h_\mu. \tag{4}$$

As a nontrivial example, one may consider the standard exponential probability distribution ν^n on $\mathbf{R}^n$, with density

$$\frac{d\nu^n(x)}{dx} = \frac{1}{2^n}\, e^{-\|x\|_1}, \quad x \in \mathbf{R}^n,$$

where $\|x\|_1 = |x_1| + \dots + |x_n|$. In this case, $h_\nu = 1$, and we get

$$h_{\nu^n} \geqslant \frac{1}{2\sqrt{6}}.$$

Equivalently, for any locally Lipschitz function f on $\mathbf{R}^n$ with ν^n-median zero,

$$\int |f|\, d\nu^n \leqslant 2\sqrt{6} \int |\nabla f|\, d\nu^n. \tag{5}$$

Similar statements with the same constants remain to hold for the one-sided exponential distribution μ on the real line with density $p(x) = e^{-x}$, $x > 0$.

As for a general probability distribution μ on the line (cf. [**5**]), the isoperimetric constant admits a simple description

$$h_\mu = \operatorname{essinf} \frac{p(x)}{\min\{F(x), 1-F(x)\}},$$

where $F(x) = \mu(-\infty, x]$, $x \in \mathbf{R}$, is the associated distribution function, and p is the density of the absolutely continuous component of μ with respect to Lebesgue measure. Moreover, the property $h_\mu > 0$ amounts to saying that μ represents the transform of the exponential measure ν under a Lipschitz map $T : \mathbf{R} \to \mathbf{R}$.

Theorem 1 was obtained in [**5**] with a proof which is rather lengthy and routine especially in part concerning the induction step. In this note we consider a different approach relating Theorem 1 to Sobolev type inequalities

$$I\left(\int f\,d\mu\right) \leqslant \int \sqrt{I(f)^2 + |\nabla f|^2}\,d\mu, \tag{6}$$

where I is a given nonnegative function on $[0,1]$, and where f is an arbitrary locally Lipschitz function on M with values in $[0,1]$. Such inequalities belong to the family of additive analytic inequalities. The additivity property of (6) means that it may always be extended to the n-dimensional space (M^n, ρ_n, μ_n) to get

$$I\left(\int f\,d\mu^n\right) \leqslant \int \sqrt{I(f)^2 + |\nabla f|^2}\,d\mu^n, \tag{7}$$

once we have it for $n = 1$, i.e., for the original triple (M, ρ, μ). Moreover, on Lipschitz functions f, which approximate indicator functions 1_A of Borel subsets of M^n, (7) turns into

$$\left(\mu^n\right)^+(A) \geqslant I(\mu^n(A)).$$

Therefore, to obtain this dimension-free isoperimetric inequality, it is sufficient to establish the analytic inequality (7) in dimension one. This approach was proposed in [**6**], where it was applied to reach the Gaussian isoperimetric inequality of Sudakov and Tsirel'son [**7**] and Borell [**8**]. Adapting the hypothesis (3) to discrete gradients, one may also include various discrete models. See [**9, 10, 11, 12, 13, 14, 15**] for discussions and related results.

As we will see, this approach can also lead to Theorem 1 with (surprisingly) the same coefficient $\frac{1}{2\sqrt{6}}$ as in (4), although there is no reason to believe that it is sharp. The statement below was put in the thesis [**16**] and since then has not been yet published.

Theorem 2. *Let $I(p) = 4p(1-p)$, $0 \leqslant p \leqslant 1$, and let $C = \frac{4\sqrt{6}}{h_\mu}$. For any locally Lipschitz function $f : M^n \to [0,1]$*

$$I\left(\int f\,d\mu^n\right) \leqslant \int \sqrt{I(f)^2 + C^2|\nabla f|^2}\,d\mu^n. \tag{8}$$

By $\sqrt{a^2+b^2} \leqslant a + b$ $(a, b \geqslant 0)$, the inequality (8) implies

$$4\,\mathrm{Var}_{\mu^n}(f) = I\left(\int f\,d\mu^n\right) - \int I(f)\,d\mu^n \leqslant C \int |\nabla f|\,d\mu^n, \tag{9}$$

where

$$\mathrm{Var}_{\mu^n}(f) = \int f^2\,d\mu^n - \left(\int f\,d\mu^n\right)^2$$

stands for the variance of f under the measure μ^n. In particular, starting with an arbitrary Borel set $A \subset M^n$ and approximating the indicator function 1_A by Lipschitz f's on M^n (as in

Lemma 3.5 of [**5**]), we get from (9) that

$$\big(\mu^n\big)^+(A) \geqslant \frac{4}{C}\,\mu^n(A)(1-\mu^n(A)). \tag{10}$$

Here the right-hand side can be bounded from below by $\frac{2}{C}\,\min\{\mu^n(A), 1-\mu^n(A)\}$, and we arrive at the required inequality (4). Thus, Theorem 2 implies Theorem 1.

The proof of Theorem 2 in dimension one is based upon the following generalization of Cheeger's inequality (2), which we state below as a lemma and refer to [**5**, Theorem 3.1] for a proof.

Lemma. *Let Ψ be a Young function (i.e., an even, convex function on $\mathbf{R}$ such that $\Psi(0)=0$, $\Psi(t)>0$ for $t>0$). If $h_\mu>0$, for any locally Lipschitz function f on M,*

$$\int \Psi(f-m)\,d\mu \leqslant \int \Psi\Big(\frac{c_\Psi}{h_\mu}\,|\nabla f|\Big)\,d\mu, \tag{11}$$

where m is (any) median for f with respect μ and where $c_\Psi = \sup_{t>0}\frac{t\Psi'(t)}{\Psi(t)}$.

Note the constant c_Ψ is well defined and does not depend on the choice of the Radon–Nikodym derivative Ψ' of Ψ. The particular case $\Psi(t)=|t|$ in (11) returns us to (2) with $h=h_\mu$, while the case $\Psi(t)=|t|^2$ yields

$$\frac{h^2}{4}\,\int |f-m|^2\,d\mu \leqslant \int |\nabla f|^2\,d\mu,$$

which is a slight improvement over the Poincaré type inequality

$$\frac{h^2}{4}\,\mathrm{Var}_\mu(f) \leqslant \int |\nabla f|^2\,d\mu. \tag{12}$$

Proof of Theorem 2. Let $n=1$. Take an f on M with values in [0,1]. Subtracting $I(f)$ from both sides of (8), rewrite this inequality as

$$4\,\mathrm{Var}_\mu(f) \leqslant \int \Big[\sqrt{I(f)^2 + C^2|\nabla f|^2} - I(f)\Big]\,d\mu.$$

Since $u=I(f)$ is bounded by 1 and since the functions $u\to\sqrt{u^2+v^2}-u$ are nonincreasing in $u\geqslant 0$, it is sufficient to show that

$$4\,\mathrm{Var}_\mu(f) \leqslant \int \Big[\sqrt{1+C^2|\nabla f|^2} - 1\Big]\,d\mu. \tag{13}$$

For $|t|\leqslant 1$, there is an elementary inequality $4t^2 \leqslant \sqrt{1+24\,t^2}-1$. Applying it with $t=f-m$ and using Lemma 1 with the Young function $\Psi(t)=\sqrt{1+t^2}-1$, in which case $c_\Psi=2$, we get

$$4\,\mathrm{Var}_\mu(f) \leqslant 4\int (f-m)^2\,d\mu \leqslant \int \Psi\Big(\sqrt{24}\,(f-m)\Big)\,d\mu \leqslant \int \Psi\bigg(\frac{2\sqrt{24}}{h_\mu}\,|\nabla f|\bigg)\,d\mu.$$

Thus, we have arrived at the desired estimate (13) with

$$C = \frac{2\sqrt{24}}{h_\mu} = \frac{4\sqrt{6}}{h_\mu}.$$

Corollary. *Let ν_n be the standard exponential measure on $\mathbf{R}^n$. For any locally Lipschitz function f on $\mathbf{R}^n$ with values in* $[0,1]$

$$\mathrm{Var}_{\nu^n}(f) \leqslant \sqrt{6} \int |\nabla f| \, d\nu^n. \tag{14}$$

Indeed, Sobolev type inequalities of this form hold in the class of all locally Lipschitz f with values in [0,1], if and only if they hold in the asymptotic sense for indicator functions (cf. [**17**]). But for $f = 1_A$, (14) is reduced to the isoperimetric inequality

$$\big(\nu^n\big)^+(A) \geqslant \frac{1}{\sqrt{6}} \, \nu^n(A)(1 - \mu^n(A)), \tag{15}$$

which is a particular case of (10) for the exponential measure ν, and where $C = 4\sqrt{6}$ as in Theorem 2. But (10) was already shown to be a consequence of (8).

The inequality (14) complements and is equivalent, up to a universal factor, to the inequality (5). Note that the assumption $m(f) = 0$ in (5) can be replaced with $\mathbf{E}f = \int f \, d\nu_n = 0$. Indeed, only indicator functions are important in

$$\int |f - \mathbf{E}f| \, d\nu^n \leqslant 2\sqrt{6} \int |\nabla f| \, d\nu^n.$$

But for such functions we are again reduced to (15).

Remarks. As was already mentioned (cf. (12)), the best constant $\lambda_1 = \lambda_1(\mu)$ in the Poincaré type inequality

$$\lambda_1 \mathrm{Var}_\mu(f) \leqslant \int |\nabla f|^2 \, d\mu$$

is connected with the isoperimetric constant $h = h_\mu$ via the general relation $\lambda_1 \geqslant \frac{h^2}{4}$ (Cheeger's theorem). When a probability measure μ on $M = \mathbf{R}^d$ is absolutely continuous and has a log-concave density with respect to Lebesgue measure (i.e, the measure is log-concave), there is a converse bound $h^2 \geqslant c\lambda_1$ with some numerical constant $c > 0$. This result was obtained by Ledoux, using semi-group arguments, cf. [**12**]. On the other hand, Poincaré type inequalities are of additive type in the sense that for product measures we have $\lambda_1(\mu^n) = \lambda_1(\mu)$. Hence, for any log-concave probability measure μ on the real line,

$$h_{\mu^n}^2 \geqslant c \, \lambda_1(\mu) \geqslant \frac{c}{4} \, h_\mu^2.$$

This leads to Theorem 1 with the coefficient $\sqrt{c}/2$. In particular, the argument may be applied to the exponential measures ν^n.

In the general case, the quantities $\lambda_1(\mu)$ and h_μ are, however, not equivalent. Nevertheless, the previous argument may still be used to cover the general metric case as in Theorem 1 by applying one comparison property, which was found by F. Barthe. Namely, for a symmetric log-concave probability measure ν on the line, introduce the associated isoperimetric functions for product measures ν^n on $\mathbf{R}^n$,

$$I_{\nu^n}(p) = \inf\big\{(\nu^n)^+(B) : \nu^n(B) = p, \quad B \subset \mathbf{R}^n \text{ (Borel)}\big\}.$$

According to [**13**, Theorem 10], applied to product measures with equal marginals, if a Borel probability measure μ on the metric space (M,d) satisfies an isoperimetric inequality

$$\mu^+(A) \geqslant I_\nu(\mu(A)), \quad A \subset M,$$

then we have a similar isoperimetric inequality for all product spaces (M^n, μ^n),

$$(\mu^n)^+(A) \geqslant I_{\nu^n}(\mu(A)), \quad A \subset M^n.$$

In other words, there is a comparison property $I_\mu \geqslant I_\nu \Rightarrow I_{\mu^n} \geqslant I_{\nu^n}$ in terms of the isoperimetric functions.

In particular, this property may be applied to the standard exponential measure ν on the real line. In this case (cf. [**18**, Propositions 2.1-2.2], it is known that $I_\nu(p) = \min\{p, 1-p\}$, for all $p \in [0,1]$. So, the combination of the Ledoux and Barthe theorems leads again to Theorem 1 with the coefficient $\sqrt{c}/2$, as above.

Acknowledgments. I would like to thank Franck Barthe for reading preprint and valuable comments.

This work was partially supported by NSF (grant No. DMS-0706860).

References

1. J. A. Cheeger, "A lower bound for the smallest eigenvalue of the Laplacian. Problems in analysis" In: *Papers Dedicated to Salomon Bochner, 1969*, Princeton Univ. Press, Princeton (1970), pp. 195-199.
2. V. G. Maz'ya, "Classes of domains and imbedding theorems for function spaces" [in Russian], *Dokl. Akad. Nauk SSSR* **133**, 527-530 (1960); English transl.: *Soviet Math. Dokl.* **1** 882-885 (1960).
3. V. G. Maz'ya, *Sobolev Spaces*, Springer-Verlag, Berlin-New York (1985).
4. A. Grigor'yan, "Isoperimetric inequalities and capacities on Riemannian manifolds" In: *The Maz'ya Anniversary Collection, Vol. 1 (Rostock, 1998)*, Birkhauser, Basel (1999), pp. 139-153.
5. S. G. Bobkov and C. Houdré, "Isoperimetric constants for product probability measures," *Ann. Probab.* **25**, No. 1, 184-205 (1997).
6. S. G. Bobkov, "An isoperimetric inequality on the discrete cube, and an elementary proof of the isoperimetric inequality in Gauss space," *Ann. Probab.* **25**, No. 1, 206-214 (1997).
7. V. N. Sudakov and B. Tsirel'son, "Extremal properties of half-spaces for spherically invariant measures" [in Russian] *Zap. Nauchn. Sem. LOMI* **41**, 14-24 (1974); English transl.: *J. Soviet Math. 9*, 9-18 (1978).
8. C. Borell, "The Brunn–Minkowski inequality in Gauss space," *Invent. Math.* **30**, 207-216 (1975).
9. D. Bakry and M. Ledoux, "Lévy-Gromov isoperimetric inequality for an infinite dimensional diffusion generator," Invent. Math. *Invent. Math.* **123**, 259-281 (1996).
10. S. G. Bobkov and F. Götze, "Discrete isoperimetric and Poincaré type inequalities," *Probab. Theory Rel. Fields* **114**, 245-277 (1999).
11. M. Ledoux, "Concentration of measure and logarithmic Sobolev inequalities" In: *Lect. Notes Math.* **1709**, Springer (1999), pp. 120-216.
12. M. Ledoux, "Spectral gap, logarithmic Sobolev constant, and geometric bounds," In: *Surv. Differ. Geom.* **9** Int. Press, Somerville (2004), p. 219-240.
13. F. Barthe, "Log-concave and spherical models in isoperimetry," *Geom. Funct. Anal.* **12**, No. 1, 32-55 (2002).
14. F. Barthe and B. Maurey, "Some remarks on isoperimetry of Gaussian type," *Ann. Inst. H. Poincaré Probab. Statist.* **36**, No. 4, 419-434 (2000).

15. B. Zegarlinski, "Isoperimetry for Gibbs measures," *Ann. Probab.* **29**, No. 2, 802-819 (2001).
16. S. G. Bobkov, *Isoperimetric Problems in the Theory of Infinite Dimensional Probability Distributions*, Doct. Dissert. St-Petersburg University, 1997.
17. S. G. Bobkov and C. Houdré, "Some connections between isoperimetric and Sobolev type inequalities," *Memoirs Am. Math. Soc.* **129**, No. 616 (1997).
18. S. G. Bobkov, "Extremal properties of half-spaces for log-concave distributions," *Ann. Probab.* **24**, No. 1, 35-48 (1996).

Submitted date: April 6, 2009