Hardy Type Inequalities via Riccati and Sturm–Liouville Equations

Sergey Bobkov and Friedrich Götze

Abstract We discuss integral estimates for domain of solutions to some canonical Riccati and Sturm–Liouville equations on the line. The approach is applied to Hardy and Poincaré type inequalities with weights.

1 Introduction

Given a function $V = V(t)$ in $t \geqslant 0$, consider the Riccati equation

$$y'(t) = y(t)^2 + V(t) \tag{1.1}$$

with initial condition

$$y(0) = 0. \tag{1.2}$$

A standard question about (1.1)–(1.2) is how to exactly determine or to estimate in terms of V the length of the maximal interval $[0, t_0)$, $t_0 > 0$, on which a (unique) solution y exists. Known results on estimates for t_0 usually treat more general Cauchy's problems, and being applied to the above special situation, they depend upon the growth of the maximum of $|V|$ on intervals $[0, t]$ with growing t. Throughout the paper, we assume that V is nonnegative, continuous, and is not identically zero. In this case, an important information can be derived by applying suitable comparison arguments, which lead to more sensitive integrable estimates. In particular, we prove the following

Sergey Bobkov
University of Minnesota, Minneapolis, MN 55455, USA, e-mail: bobkov@math.umn.edu

Friedrich Götze
Bielefeld University, Bielefeld 33501, Germany, e-mail: goetze@math.uni-bielefeld.de

V. Maz'ya (ed.), *Sobolev Spaces in Mathematics I,*
International Mathematical Series.
doi: 10.1007/978-0-387-85648-3,

Theorem 1.1. *Define*

$$\overline{V}(t) = \int_0^t V(u)\,du, \quad t \geqslant 0.$$

The maximal value of t_0 *satisfies*

$$\frac{1}{4t_0} \leqslant \sup_{0<s<1} \left[(1-s)\overline{V}(t_0 s)\right] \leqslant \frac{1}{t_0}. \tag{1.3}$$

For example, for $V(t) = t^{\alpha-1}$, $\alpha \geqslant 1$, this gives

$$\frac{1}{4^{1/(\alpha+1)}}\,\frac{\alpha+1}{\alpha^{(\alpha-1)/(\alpha+1)}} \leqslant t_0 \leqslant \frac{\alpha+1}{\alpha^{(\alpha-1)/(\alpha+1)}}.$$

In particular, $t_0 \to 1$ as $\alpha \to +\infty$.

Transforming (1.1) to second order linear differential equations, one may give an equivalent formulation of Theorem 1.1 as a statement about the first eigenvalue λ_0 for the regular Sturm–Liouville equation

$$\frac{d}{dt}\left(q(t)\frac{d}{dt}z(t)\right) = \lambda p(t)z(t), \quad a \leqslant t \leqslant b, \tag{1.4}$$

with boundary conditions $z(a) = z'(b) = 0$. Introduce the quantity

$$A(p,q) = \sup_{a<x<b} \left[\int_a^x \frac{1}{q(t)}\,dt \int_x^b p(t)\,dt\right].$$

Theorem 1.2. *For all positive continuous functions* p *and* q *on* $[a,b]$

$$A(p,q) \leqslant \frac{1}{\lambda_0} \leqslant 4A(p,q). \tag{1.5}$$

We consider the estimates (1.3) and (1.5) as another approach, from differential equation point of view, to a result, obtained by Kac and Krein [7] in 1959 and later by Artola [1], Talenti [14], and Tomaselli [15], about Hardy type inequalities with weights. In these inequalities, one tries to determine or estimate the best constant $C = C(p,q)$ satisfying

$$\int_a^b f(x)^2 p(x)\,dx \leqslant C \int_a^b f'(x)^2 q(x)\,dx, \tag{1.6}$$

where f is an arbitrary absolutely continuous function on $[a,b)$ such that $f(a) = 0$. Their result, including the case $b = +\infty$ as well, asserts that

$$A(p,q) \leqslant C(p,q) \leqslant 4A(p,q) \tag{1.7}$$

(actually, they treated a more general L^α-norm in (1.6)). In 1972, Muckenhoupt [11] gave a complete account on this result and extended it to arbitrary positive measures in place of $p(x)dx$ and $q(x)dx$. In general (when the interval is unbounded), it might occur that $A(p,q)$ is infinite. The property $A(p,q) < +\infty$ is sometimes called the *Muckenhoupt condition*, although the two-sided inequality (1.7) is associated with it, as well. For more general results and references we refer the interested reader to the monograph [9]. The connection of (1.6) with (1.4) is as follows: in the regular case, the extremal functions in Hardy type inequalities exist and satisfy the boundary value problem of Theorem 1.2 with the smallest possible value of λ. In particular, $C(p,q) = 1/\lambda_0$. It should also be clear that, in (1.6) and (1.7), the regular case easily implies the general case, where p and q are defined on the half-axis $(a, +\infty)$.

In a more rigorous manner, we consider the corresponding variational problem in Sect. 4, where Theorem 1.2 is proved and is shown to imply (1.7). In Sect. 2, we prove Theorem 1.1 and some related statements. In Sect. 3, we consider a particular case of Theorem 1.2 with $q \equiv 1$. We finish the paper in Section 5, where we derive an analogue of (1.5) for the boundary conditions $z'(a) = z'(b) = 0$. These conditions turn out to be connected with another important family of inequalities of Poincaré type. The reader may find some results connecting Hardy type inequalities with weights with Poincaré and logarithmic Sobolev inequalities in [2], where Muckenhoupt's characterization was essentially used (see also [10] for discrete analogues).

Theorems 1.1 and 1.2 can easily be extended to more general equations such as $y'(t) = y(t)^\beta + V(t)$ and $(qz'(t)^\beta)' = -\lambda p(t) z^\beta$ respectively. We will not study these equations in order to make easy the presentation of main techniques in the basic case $\alpha = 2$. It should however be noted that these are precisely the equations which are needed for studying the Hardy type inequalities (1.6) with respect to the norms in general Lebesgue spaces (rather than in L^2).

2 Riccati Equations

At first, it is convenient to consider the Riccati equation (1.1) in the semi-open interval $[0,1)$ and assume that V is defined, is nonnegative and continuous on this interval (and is not identically zero). One is looking for some conditions, necessary and sufficient, which would guarantee the existence of a solution to (1.1) on the whole interval $[0,1)$. If it exists (and is thus unique), it should necessarily belong to the class $C^1[0,1)$ of all continuously differentiable functions on $[0,1)$. Introducing the integral operator

$$Af(t) = \int_0^t f(u)^2\,du + \overline{V}(t), \qquad 0 \leqslant t < 1,$$

we may reformulate our task as a problem on the existence of a solution $y = y(t)$ in $C^1[0,1)$ to the nonlinear integral equation

$$Ay = y \tag{2.1}$$

under the initial condition $y(0) = 0$.

A canonical way to construct a solution to the Cauchy problem $y'(t) = \Psi(t, y(t))$ and, in particular, to the problem (1.1), where $\Psi(t,y) = V(t) + y^2$, is to start from a function y_0, recursively defining the sequence

$$y_1 = Ay_0,\ y_2 = Ay_1,\ \ldots,\ y_{n+1} = Ay_n, \quad n \geqslant 0.$$

Certain conditions on V guarantee the convergence of Ay_n to a solution on some interval $[0, t_1)$. One general sufficient condition for convergence (see, for example, [5]) may be formulated as follows. Consider the maximum $M = \max_D \Psi$ on the rectangle $D = [0,\alpha] \times [0,\beta]$. Then one can take

$$t_1 = \min\left\{\alpha, \frac{\beta}{M}\right\} = \min\left\{\alpha, \frac{\beta}{\|V\|_{C[0,\alpha]} + \beta^2}\right\},$$

where $\|V\|_{C[0,\alpha]} = \max_{0 \leqslant t \leqslant \alpha} V(t)$. Optimizing over β so that to maximize t_1, we arrive at

$$t_1(V) = \sup_{0<\alpha<1} \min\left\{\alpha, \frac{1}{2}\ \|V\|_{C[0,\alpha]}^{-1/2}\right\}.$$

Although choosing some other domains D may improve this value t_1 for concrete V, we are in a typical situation where one has to require the boundedness of V on $[0,1)$ in order to reach the value $t_1 = 1$. In particular, the above formula gives $t_1(\lambda V) = 1$ only if $\lambda \leqslant 1/(4 \sup V)$.

Now let us look at the convergence of Ay_n by using some comparison arguments and first derive the following

Lemma 2.1. *A solution to* (2.1) *under the initial condition* $y(0) = 0$ *exists if and only if for some nonnegative measurable function* f *on* $[0,1)$ *for all* $t \in [0,1)$

$$f(t) \geqslant \overline{V}(t) \quad \textit{and} \quad Af(t) \leqslant f(t). \tag{2.2}$$

Proof. Clearly, if y is a solution, then $f = y$ satisfies (2.2). To prove the converse, we assume that f satisfies (2.2). We start from $y_0 \equiv 0$ and define a sequence y_n as above. In particular, $y_1 = \overline{V}$. We set

$$y(t) = \sup_n y_n(t).$$

Note that the operator A is monotone: if $0 \leqslant g_1 \leqslant g_2$, then $0 \leqslant Ag_1 \leqslant Ag_2$. Since $\overline{V} \leqslant f$, we get $y_2 = A\overline{V} \leqslant Af \leqslant f$. Repeating the argument (i.e., by induction), we see that $y_n \leqslant f$, for all n. Therefore, the function y is finite, measurable, and satisfies $y \leqslant f$. In addition, for all n, $Ay \geqslant Ay_n = y_{n+1}$, so that $Ay \geqslant y$. On the other hand, the sequence y_n is nondecreasing: $y_2 = Ay_1 \geqslant Ay_0 = y_1$, $y_3 = Ay_2 \geqslant Ay_1 = y_2$, and so on. Thus, $y_n(t) \uparrow y(t)$ as $n \to \infty$. Hence, by the Tonneli monotone convergence theorem, $Ay_n(t) \uparrow Ay(t)$ for all $t \in [0,1)$. Taking the limit in $Ay_n = y_{n+1} \leqslant y$, we conclude that $Ay \leqslant y$. The two estimates give $Ay = y$.

The function of the form Ay, as soon as it is finite, must be absolutely continuous. Hence y is absolutely continuous, and this implies that y is in $C^1[0,1)$. □

Remark 2.2. According to the above proof, we may add to Lemma 2.1 another characterization. Consider a pointwise limit of the nondecreasing sequence

$$y_V(t) = \lim_{n\to\infty} [\underbrace{AA\ldots A}_{n \text{ times}}\, y_0](t), \quad y_0(t) \equiv 0,$$

which might be finite or not. Then the existence on the interval [0,1) of a solution y to (2.1) under the initial condition (1.2) is equivalent to the property that $y_V(t) < +\infty$, for all $t \in [0,1)$. In this case, $y = y_V$ provides the solution.

In particular, since from $0 \leqslant V \leqslant W$ it follows that $y_V \leqslant y_W$, the existence of a solution to (1.1) with a function W implies the existence of a solution to (1.1) with any (continuous) function $V \leqslant W$. Such a comparison property was given by Levin [8], who considered even more general situation, where V is not necessarily nonnegative, but still satisfies $|V| \leqslant W$.

The above reformulation also holds when we consider the Riccati equation on a larger interval or the whole half-axis $[0,+\infty)$. Then $t_0 = \sup\{t \geqslant 0 : y_V(t) < +\infty\}$.

We need the following assertion.

Lemma 2.3. *Any solution y to the Riccati equation* (1.1) *under the initial condition $y(0) = 0$ satisfies for all $t \in [0,1)$*

$$y(t) < \frac{1}{1-t}.$$

Proof. Set $t_1 = \max\{t \in [0,1) : y(t) = 0\}$. Since y must be nondecreasing and V is not identically zero, the point t_1 is well defined and lies in $[0,1)$. For $t \in (t_1,1)$ we have $y'(t) \geqslant y(t)^2 > 0$, which implies that the function $g(t) = \frac{1}{y(t)} + t$ decreases in $(t_1,1)$. In particular, $g(t) > g(1-) \geqslant 1$. □

Now, we are ready to estimate the supremum $\lambda(V)$ of all $\lambda \geqslant 0$, for which there exists a solution $y = y(t)$ to the Riccati equation

$$y'(t) = y(t)^2 + \lambda V(t), \quad 0 \leqslant t < 1, \tag{2.3}$$

under the same initial condition $y(0) = 0$. As a consequence of the two lemmas, we derive the following statement closely related to Theorem 1.1.

Theorem 2.4. *We have*

$$\sup_{0<t<1} \left[(1-t)\overline{V}(t)\right] \leqslant \frac{1}{\lambda(V)} \leqslant 4 \sup_{0<t<1} \left[(1-t)\overline{V}(t)\right]. \tag{2.4}$$

In particular, $\lambda(V) > 0$ if and only if $\overline{V}(t) = O(\frac{1}{1-t})$ as $t \to 1$.

Proof. First, assume that y satisfies (2.3) with $y(0) = 0$. By Lemma 2.3, $\frac{1}{1-t} > y(t)$ in $[0,1)$. Since $y'(t) \geqslant \lambda V(t)$, we also have $y(t) \geqslant \lambda\overline{V}(t)$. Hence

$$\frac{1}{1-t} > \lambda\overline{V}(t).$$

This gives the first inequality in (2.4). To prove the second one, we use Lemma 2.1 with respect to the function λV. Take any $\lambda \geqslant 0$ such that

$$\frac{1}{\lambda} \geqslant 4 \sup_{0<t<1} (1-t)\overline{V}(t),$$

so that

$$\lambda\overline{V}(t) \leqslant \frac{1}{4(1-t)}, \qquad 0 \leqslant t < 1.$$

Then for the function $f(t) = \frac{1}{2(1-t)}$ we get

$$Af(t) \equiv \int_0^t f(u)^2\,du + \lambda\overline{V}(t) = \frac{t}{4(1-t)} + \lambda\overline{V}(t)$$
$$\leqslant \frac{t}{4(1-t)} + \frac{1}{4(1-t)} \leqslant \frac{1}{2(1-t)} = f(t).$$

Thus, $Af(t) \leqslant f(t)$. On the other hand, $f(t) \geqslant \lambda\overline{V}(t)$, so the sufficient conditions of Lemma 2.1 are satisfied. Hence there is a solution y to (2.3) with $y(0) = 0$. This gives the second inequality in (2.4). Hence Theorem 2.4 is proved. □

Proof of Theorem 1.1. It remains to explain why Theorem 1.1 is an immediate consequence of Theorem 2.4. Considering (1.1) on a finite interval $[0, t_1)$ and introducing the functions $z(s) = t_1 y(t_1 s)$, $0 \leqslant s < 1$, we arrive at the Riccati equation on $[0,1)$

$$z'(s) = z(s)^2 + t_1^2 V(t_1 s) \tag{2.5}$$

under the same initial condition $z(0) = 0$. Now, apply Theorem 2.4 to $V_{t_1}(s) = V(t_1 s)$. The existence of a solution z to (2.5) implies that

$$\frac{1}{t_1^2} \geqslant \frac{1}{\lambda(V_{t_1})} \geqslant \sup_{0<s<1} \left[(1-s)\overline{V}_{t_1}(s)\right] = \frac{1}{t_1} \sup_{0<s<1} \left[(1-s)\overline{V}(t_1 s)\right].$$

This leads to the second inequality in (1.3) for any t_1 such that (1.1) has a solution on $[0, t_1)$ with the initial condition (1.2). Let t_0 denote the maximal value t_1 with this property. By the second inequality in (2.4), a solution z to the equation (2.5) exists on $[0, 1)$ if

$$\frac{1}{t_1^2} \geqslant 4 \sup_{0<s<1} [(1-s)\overline{V}_{t_1}(s)],$$

i.e., if

$$\frac{1}{t_1} \geqslant 4 \sup_{0<s<1} [(1-s)\overline{V}(t_1 s)]. \tag{2.6}$$

The right-hand side of (2.6) is nondecreasing and continuous in $t_1 > 0$, so there exists a unique point t_2 which turns this inequality into equality; moreover, for $t > t_2$, we have the converse inequality

$$\frac{1}{t} < 4 \sup_{0<s<1} \left[(1-s)\overline{V}(ts)\right].$$

Since $t_0 \geqslant t_2$, we thus obtain the left inequality in (1.3) and Theorem 1.1 follows. $\square$

3 Transition to Sturm–Liouville Equations

One may equivalently reformulate Theorem 1.1 as a statement about the first zero of solutions to a second order differential equation. Here, we consider only the simplest equation

$$z''(t) = -V(t)z(t), \quad t \geqslant 0, \tag{3.1}$$

under the initial conditions

$$z(0) = 1, \quad z'(0) = 0 \tag{3.2}$$

(the condition $z(0) = 1$ has a matter of normalization, only). As in Theorem 1.1, assume that V is a nonnegative continuous function on $[0, +\infty)$, which is not identically zero. It is well known (see, for example, [12]) that any second order linear differential equation with continuous coefficients and given initial conditions has a unique nontrivial solution. Moreover, on every finite interval, the solution has a finite number of zeros. In the case of (3.1), (3.2) with $V \geqslant 0$

and $V \neq 0$, we may define

$$t_0 = \min\{t > 0 : z(t) = 0\},$$

and one would like to estimate t_0. Since $z(0) > 0$, the function z must be positive on $[0, t_0)$, and we may introduce a new function

$$y(t) = -\frac{z'(t)}{z(t)}, \quad 0 \leqslant t < t_0.$$

It satisfies the Riccati equation (1.1) with initial condition (1.2). Conversely, starting from a function y satisfying (1.1), (1.2) on $[0, t_0)$, one may define the function $z(t) = \exp\{-\int_0^t y(s)ds\}$, which will satisfy (3.1), (3.2) on the same interval. Thus, we may conclude:

Corollary 3.1. *The minimal zero t_0 of the solution z to the problem* (3.1), (3.2) *satisfies*

$$\frac{1}{4t_0} \leqslant \sup_{0<s<1} \left[(1-s)\overline{V}(t_0 s)\right] \leqslant \frac{1}{t_0}.$$

Similarly, we have an equivalent analogue of Theorem 2.4. Assume that V is now defined on $[0, 1)$, is continuous, nonnegative and is not identically zero. Consider in $[0, 1)$ the equation

$$z''(t) = -\lambda V(t) z(t). \tag{3.3}$$

Corollary 3.2. *Let $\lambda(V)$ be the supremum of all $\lambda \geqslant 0$, for which a solution z to the problem* (3.2), (3.3) *is positive in* $[0, 1)$*. Then*

$$\sup_{0<t<1} \left[(1-t)\overline{V}(t)\right] \leqslant \frac{1}{\lambda(V)} \leqslant 4 \sup_{0<t<1} \left[(1-t)\overline{V}(t)\right]. \tag{3.4}$$

If the limit $V(1-) = \lim_{t\to 1-0} V(t)$ exists and is finite, i.e., V is continuous on [0,1], the solutions z_λ to (3.2), (3.3) exist on the whole interval [0,1]. In particular, this is true for $\lambda = \lambda(V)$, and moreover, $z_{\lambda(V)}$ is still positive on $[0, 1)$. Indeed, z_λ depends continuously on λ, and in particular, for all $t \in [0, 1]$, $z_\lambda(t) \to z_{\lambda(V)}(t)$ as $\lambda \to \lambda(V)-$. But the functions z_λ are concave on $[0, 1]$ and satisfy $z_\lambda(0) = 1$, $z_\lambda(1) \geqslant 0$, so $z_{\lambda(V)}$ possesses the same properties. Thus, the supremum in Corollary 3.2 is actually the maximum, and a similar observation applies to Theorem 2.4.

In fact, $z_{\lambda(V)}(1) = 0$ since otherwise we would get, by continuity, that $z_\lambda(1) > 0$, for some $\lambda > \lambda(V)$, which contradicts to the maximality of $\lambda(V)$. Consequently, provided (3.2) holds, the following two conditions uniquely determine the value $\lambda = \lambda(V)$: z_λ is nonnegative and satisfies $z_\lambda(1) = 0$.

If V is additionally everywhere positive, one can further specify $\lambda(V)$ as the smallest eigenvalue λ_0 to the problem (3.2), (3.3) with boundary condition $z(1) = 0$. Indeed (see, for example, [3, 13]), in the regular case, the boundary value problem on [0,1]

$$z''(t) = -\lambda V(t)z(t), \qquad z'(0) = z(1) = 0, \tag{3.5}$$

has an infinite sequence $\lambda_0 < \lambda_1 < \dots$ of eigenvalues, and the corresponding eigenfunctions z_n have exactly n zeros in (0,1). Therefore, among these eigenfunctions and up to a constant, only z_0 does not vanish in (0,1). Getting rid of the normalization condition $z(0) = 1$, we may conclude the following:

Corollary 3.3. *Let V be continuous and positive on $[0,1]$. Then the value $\lambda(V)$ is the smallest eigenvalue λ_0 for the boundary value problem* (3.5)*. In particular, λ_0 admits the estimates* (3.4)*.*

4 Hardy Type Inequalities with Weights

As mentioned before, one may arrive at Sturm–Liouville equations starting from Hardy type inequalities with weights. Here, we show how to treat the constants in such inequalities using Corollary 3.3. To this end, consider the functional

$$J(f) = \frac{\int\limits_a^b f'(x)^2 q(x)\, dx}{\int\limits_a^b f(x)^2 p(x)\, dx},$$

where p and q are positive continuous functions on a finite interval $[a, b]$.

We denote by $W_1^2 = W_1^2[a, b]$ the Sobolev space of all absolutely continuous functions f on $[a, b]$ with square integrable (Radon–Nikodym) derivatives so that $J(f)$ is well defined for such functions provided that $f \neq 0$ (identically).

Lemma 4.1. *There exists a function f in W_1^2, $f \neq 0$, unique up to a constant, where the functional J attains its minimum within W_1^2 under the restriction $f(a) = 0$.*

The statement is well known (see, for example, [6] for related results). For the sake of completeness, we include a proof of the following assertion.

Theorem 4.2. *The quantity* $\min\{J(f) : f \in W_1^2, f \neq 0, f(a) = 0\}$ *represents the unique number $\lambda > 0$ such that the Sturm–Liouville equation*

$$(f'q)' = -\lambda f p \tag{4.1}$$

has a nontrivial nonnegative monotone solution on $[a, b]$ *with boundary conditions*

$$f(a) = f'(b) = 0. \tag{4.2}$$

Thus, it is equal to the smallest eigenvalue for this boundary value problem.

The argument consists of two parts.

Lemma 4.3. *Assume that a function* f *in* W_1^2, $f \neq 0$, *minimizes* J *on* W_1^2 *under the restriction* $f(a) = 0$. *Then the derivative* f' *may be modified on a set of Lebesgue measure zero such that the following properties are fulfilled*:

1) $f \in C^1[a, b]$;
2) f *is monotone, and moreover,* $f'(x) \neq 0$, *for all* $x \in [a, b)$;
3) $f'(b) = 0$;
4) $f'q \in C^1[a, b]$, *and Equation* (4.1) *holds.*

Proof. First note that, since $f(a) = 0$ and $f \neq 0$, we have

$$\int_a^b f(x)^2 p(x)\, dx > 0 \quad \text{and} \quad \int_a^b f'(x)^2 q(x)\, dx > 0.$$

Now, we take an arbitrary $h \in W_1^2$ with $h(a) = 0$ and consider for small ε the functions $f_\varepsilon = f + \varepsilon h$. By the Taylor expansion, as $\varepsilon \to 0$,

$$J(f_\varepsilon) = J(f) \left[1 + 2\varepsilon \left(\frac{\int_a^b f'(x)h'(x)q(x)\, dx}{\int_a^b f'(x)^2 q(x)\, dx} - \frac{\int_a^b f(x)h(x)p(x)\, dx}{\int_a^b f(x)^2 p(x)\, dx} \right) + O(\varepsilon^2) \right].$$

Since $J(f_\varepsilon) \leqslant J(f)$, the expression in the round brackets must be zero, i.e.,

$$\int_a^b (f'(x)q(x))\, h'(x)\, dx = \lambda \int_a^b (f(x)p(x))\, h(x)\, dx.$$

Using

$$h(x) = \int_a^x h'(t)\, dt, \quad a \leqslant x \leqslant b,$$

we rewrite the above expression as

$$\int_a^b (f'(x)q(x))\, h'(x)\, dx = \int_a^b \left(\lambda \int_x^b f(t)p(t)\, dt \right) h'(x)\, dx.$$

Since h' may be arbitrary in $L^2(a, b)$, we conclude that for almost all $x \in (a, b)$

$$f'(x)q(x) = \lambda \int_x^b f(t)p(t)\,dt. \tag{4.3}$$

This equality may be regarded as a definition of f'. Thus, we may assume that (4.3) holds for all $x \in [a, b]$, and as a result, immediately obtain properties 1), 3), and 4).

To get 2), we consider the function

$$g(x) = \int_a^x |f'(t)|\,dt.$$

Then $g(a) = 0$ and $g'(x) = |f'(x)|$ for almost all $x \in (a, b)$, so that

$$\int_a^b g'(x)^2 q(x)\,dx = \int_a^b f'(x)^2 q(x)\,dx.$$

We also have $g(x) \geqslant |f(x)|$ for all $[a, b]$, which implies

$$\int_a^b g(x)^2 p(x)\,dx \geqslant \int_a^b f(x)^2 p(x)\,dx$$

with equality possible only when $g(x) = |f(x)|$, for all $x \in [a, b]$ (since both g and f are continuous). This must be indeed the case since, otherwise, $J(g) < J(f)$ contradicts the basic assumption on f. Hence either $f' \geqslant 0$ almost everywhere or $f' \leqslant 0$ almost everywhere, and thus f is monotone.

Assume that $f' \geqslant 0$ almost everywhere, and thus $f' \geqslant 0$ everywhere by the continuity of f'. Since $f(a) = 0$ and $f \neq 0$ (identically), we get $f(b) > 0$. Hence f must be positive at least in a neighborhood of b, and this yields that the right-hand side of (4.3) is positive whenever $a \leqslant x < b$. Hence $f'(x) > 0$ on (a, b) according to (4.3). Lemma 4.3 follows. □

Lemma 4.4. *Given $\lambda > 0$, assume that the boundary value problem* (4.1), (4.2) *has a nontrivial monotone solution. Then for all $f \in W_1^2$, $f \neq 0$, we have $J(f) \geqslant \lambda$.*

The assumption about monotonicity is necessary. For example, in the case $p \equiv q \equiv 1$, on the interval $[0, 3\pi/2]$ there is solution $f(x) = \sin x$ to (4.1) with $\lambda = 1$, which satisfies the boundary conditions (4.2). However, the infimum of J is attained at $\psi(x) = \sin(x/3)$ and is equal to $1/9$.

Proof. The argument is not new; it was used, in particular, in [4]. Let ψ be a nontrivial nondecreasing solution. In particular, $\psi \in C^1[a,b]$, $\psi' q \in C^1[a,b]$, and ψ satisfies (4.1), (4.2). Integrating (4.1) over the interval (x,b) and using $\psi'(b) = 0$, we obtain (4.3) for ψ,

$$\psi'(x)q(x) = \lambda \int_x^b \psi(t)p(t)\,dt, \quad a \leqslant x \leqslant b. \tag{4.4}$$

Arguing as above, since $\psi(a) = 0$ and $\psi \neq 0$ (identically), we get $\psi(b) > 0$, and so ψ must be positive at least in a neighborhood of b. According to (4.4), we get $\psi'(x) > 0$ whenever $a \leqslant x < b$.

Now, we take an arbitrary f in W_1^2 with $f(a) = 0$ and write for $x \in (a,b)$

$$f(x) = \int_a^x f'(t)\,dt = \int_a^x \frac{f'(t)}{\sqrt{\psi'(t)}}\,\sqrt{\psi'(t)}\,dt,$$

so that, by the Schwarz inequality,

$$f(x)^2 \leqslant \int_a^x \frac{f'(t)^2}{\psi'(t)}\,dt \int_a^x \psi'(t)\,dt = \int_a^x \frac{f'(t)^2}{\psi'(t)}\,dt\ \psi(x).$$

Hence, by (4.4),

$$\begin{aligned}\lambda \int_a^b f(x)^2 p(x)\,dx &\leqslant \lambda \int_a^b \left(\int_a^x \frac{f'(t)^2}{\psi'(t)}\,dt\ \psi(x) \right) p(x)\,dx \\ &= \int_a^b \frac{f'(t)^2}{\psi'(t)} \left(\lambda \int_t^b \psi(x)p(x)\,dx \right) dt \ = \int_a^b f'(t)^2 q(t)\,dt.\end{aligned}$$

The proof is complete. □

Proof of Theorem 4.2. We combine Lemmas 4.3 and 4.4 (recalling an argument before Corollary 3.3 about zeros of eigenfunctions). □

Now, let us state a certain duality between Hardy type inequalities.

Lemma 4.5. *For every $c > 0$ the following two inequalities are equivalent*:

$$c \int_a^b f^2\, p\ dx \leqslant \int_a^b f'^2\, q\ dx, \quad \textit{for all } f \in W_1^2 \textit{ with } f(a) = 0; \tag{4.5}$$

$$c\int_a^b f^2/q\,dx \leqslant \int_a^b f'^2/p\,dx, \quad \textit{for all } f \in W_1^2 \textit{ with } f(b)=0. \tag{4.6}$$

In particular, the optimal constants c in (4.5) *and* (4.6) *coincide.*

Proof. It suffices to show that (4.5) implies (4.6). Denote by λ an optimal constant in (4.5) so that $c \leqslant \lambda$. By Theorem 4.2, there is a nonzero monotone function $\psi \in C^1[a,b]$ such that $\psi' q \in C^1[a,b]$, and ψ satisfies the equation $(\psi' q)' = -\lambda\psi p$ with boundary conditions $\psi(a) = \psi'(b) = 0$. In particular, the equality (4.4) holds. Define

$$y(x) = \int_x^b \psi(t)p(t)\,dt, \quad a \leqslant x \leqslant b.$$

This function is monotone, belongs to $C^1[a,b]$, and satisfies the boundary conditions $y(b) = 0$ and $y'(a) = 0$. In addition, $y'/p = -\psi$ belongs to $C^1[a,b]$. Moreover, the equality (4.4) can be rewritten in terms of y as

$$\Big(\frac{y'}{p}\Big)' = -\lambda\,\frac{y}{q},$$

which is again a Sturm–Liouville equation with respect to the functions $1/q$ and $1/p$ (in place of previous p and q). By Theorem 4.2, we conclude that (4.6) holds with constant λ in place c. □

As a consequence, we obtain Theorem 1.2 and the following assertion.

Corollary 4.6. *The smallest constant C such that the inequality*

$$\int_a^b f(x)^2 p(x)\,dx \leqslant C\int_a^b f'(x)^2 q(x)\,dx \tag{4.7}$$

holds for all f in W_1^2 with $f(a)=0$, satisfies

$$A(p,q) \leqslant C \leqslant 4A(p,q). \tag{4.8}$$

Recall that

$$A(p,q) = \sup_{a<x<b}\Big[\int_a^x \frac{1}{q(t)}\,dt \int_x^b p(t)\,dt\Big].$$

Proof of Theorem 1.2 *and Corollary* 4.6*.* We use Lemma 4.5. Without loss of generality, we assume that

$$\int_a^b p(x)\,dx = 1.$$

Introduce the distribution function

$$F(x) = \int_a^x p(t)\,dt$$

and its inverse $F^{-1} : [0,1] \to [a,b]$. Changing the variable $x = F^{-1}(t)$, we rewrite (4.5) as

$$c\int_0^1 f(F^{-1}(t))^2\,dt \leqslant \int_0^1 f'(F^{-1}(t))^2\,\frac{q(F^{-1}(t))}{p(F^{-1}(t))}\,dt.$$

In terms of $z(t) = f(F^{-1}(t))$, we again arrive at the Hardy type inequality on $[0,1]$

$$c\int_0^1 z(t)^2\,dt \leqslant \int_0^1 z'(t)^2\;p(F^{-1}(t))\,q(F^{-1}(t))\,dt$$

with boundary condition $z(0) = 0$. By Lemma 4.5, this is equivalent to

$$c\int_0^1 z(t)^2\,\frac{1}{p(F^{-1}(t))\,q(F^{-1}(t))}\,dt \;\leqslant\; \int_0^1 z'(t)^2\,dt \tag{4.9}$$

in the class of all $z \in W_2[0,1]$ such that $z(1) = 0$. Thus, the minimal constant $c = c(p,q)$ in (4.5) coincides with the optimal constant $c = c(V)$ in (4.9) on [0,1] under the restriction $z(1) = 0$ and with respect to the weight function

$$V(t) = \frac{1}{p(F^{-1}(t))\,q(F^{-1}(t))}.$$

On the other hand, by Theorem 4.2, $c(p,q)$ is the smallest eigenvalue $\lambda_0 = \lambda_0(p,q)$ for the boundary value problem (4.1), (4.2), while $c(V)$ is the smallest eigenvalue $\lambda(V)$ for the boundary value problem (3.5):

$$z'' = -\lambda V z, \quad z'(0) = z(1) = 0.$$

Hence $\lambda_0(p,q) = \lambda(V) = 1/C$, where C is the optimal constant in (4.7). By Corollary 3.3, all these quantities admit the estimates (3.4). However,

$$\sup_{0<t<1}\,(1-t)\overline{V}(t) = \sup_{0<t<1}\,(1-t)\int_0^t \frac{1}{p(F^{-1}(s))\,q(F^{-1}(s))}\,ds$$

$$= \sup_{0<t<1} (1-t) \int\limits_a^{F^{-1}(t)} \frac{1}{q(x)}\, dx$$

$$= \sup_{a<r<b} (1-F(r)) \int\limits_a^{r} \frac{1}{q(x)}\, dx$$

$$= \sup_{a<r<b} \int\limits_r^{b} p(x)\, dx \int\limits_a^{r} \frac{1}{q(x)}\, dx.$$

Therefore, (3.4) turns into (4.8). Consequently, Theorem 1.2 and Corollary 4.6 are proved. □

5 Poincaré Type Inequalities

Similarly to Theorem 1.2, we consider here the Sturm–Liouville equation

$$(f'q)' = -\lambda f p, \tag{5.1}$$

but with boundary conditions

$$f'(a) = f'(b) = 0. \tag{5.2}$$

As before, our case is regular, i.e., p and q are assumed to be positive continuous functions on a finite interval $[a, b]$. Denote by m the (unique) number in (a, b) such that $\int\limits_a^m p(x)\, dx = \int\limits_m^b p(x)\, dx$ and introduce the quantities

$$A_0 = \sup_{a<x<m} \int\limits_x^m \frac{1}{q(t)}\, dt \int\limits_a^x p(t)\, dt, \qquad A_1 = \sup_{m<x<b} \int\limits_m^x \frac{1}{q(t)}\, dt \int\limits_x^b p(t)\, dt.$$

Theorem 5.1. *The second smallest eigenvalue* λ_1 *for the boundary value problem* (5.1)-(5.2) *satisfies*

$$\frac{1}{2} \min(A_0, A_1) \leqslant \frac{1}{\lambda_1} \leqslant 4 \min(A_0, A_1).$$

Recall that the smallest eigenvalue λ_0 is zero (and corresponds to the eigenfunction $f \equiv 1$). Often, λ_1 is called the first nontrivial eigenvalue.

Proof. As in Theorem 1.2, it is well known that λ_1 represents the best constant in the Poincaré type inequality

$$\lambda_1 \int_a^b f(x)^2 p(x)\,dx \leqslant \int_a^b f'(x)^2 q(x)\,dx, \tag{5.3}$$

where f is an arbitrary function in $W_1^2[a,b]$ such that

$$\int_a^b f(x)p(x)\,dx = 0. \tag{5.4}$$

We connect (5.3) and (5.4) to Hardy type inequalities and then apply Corollary 4.6. To this end, we observe that up to an absolute factor, in front of λ_1 in (5.3), the restriction (5.4) can be replaced by

$$f(m) = 0. \tag{5.5}$$

Indeed, without loss of generality, we may assume that

$$\int_a^b p(x)\,dx = 1$$

and denote by $\mu(dx)$ the measure $p(x)\,dx$ on $[a,b]$. Then (5.3) and (5.4) can be written as

$$\lambda_1 \mathrm{Var}_\mu(f) \equiv \lambda_1 \left[\int f^2\,d\mu - \left(\int f\,d\mu\right)^2\right] \leqslant \int_a^b f'(x)^2 q(x)\,dx, \tag{5.6}$$

which holds for all f in W_1^2 without any restrictions. Hence if

$$c\int f^2\,d\mu \leqslant \int_a^b f'(x)^2 q(x)\,dx \tag{5.7}$$

holds assuming (5.5), we obtain (5.6) with $\lambda_1 = c$ since $\mathrm{Var}_\mu(f) \leqslant \int f^2\,d\mu$.

Conversely, assume that (5.6) is fulfilled for a constant λ_1. Take any function f in W_1^2 such that $f = 0$ on $[a,m]$. Then, by the Cauchy inequality,

$$\left(\int f\,d\mu\right)^2 = \left(\int f\,\mathbf{1}_{[a,m]}\,d\mu\right)^2 \leqslant \frac{1}{2}\int f^2\,d\mu,$$

where $\mathbf{1}_{[a,m]}$ denotes the characteristic function of the interval $[a, m]$. Hence $\int f^2\, d\mu \leqslant \frac{1}{2} \operatorname{Var}_\mu(f)$ and, by (5.6), we obtain (5.7) for such a function f with $c = \lambda_1/2$. The same holds when $f = 0$ on $[m, b]$. At last, just assuming (5.5), we can apply (5.7) to $f_0 = f\,\mathbf{1}_{[a,m]}$ and $f_1 = f\,\mathbf{1}_{[m,b]}$ with $c = \lambda_1/2$. Adding the two corresponding inequalities, we arrive at (5.7) for f. Thus, the optimal constants in the Poincaré type inequality (5.6) and in the Hardy type inequality (5.7) (the latter being considered under (5.5)) are connected via

$$\frac{1}{2c} \leqslant \frac{1}{\lambda_1} \leqslant \frac{1}{c}. \tag{5.8}$$

It is obvious that $c = \min(c_0, c_1)$, where c_0 and c_1 are optimal in

$$c_0 \int_a^m f(x)^2 p(x)\, dx \leqslant \int_a^m f'(x)^2 q(x)\, dx,$$

$$c_1 \int_m^b f(x)^2 p(x)\, dx \leqslant \int_m^b f'(x)^2 q(x)\, dx$$

under the restriction (5.5). Therefore, by Corollary 4.6, we have $A_0 \leqslant \frac{1}{c_0} \leqslant 4A_0$ and $A_1 \leqslant \frac{1}{c_1} \leqslant 4A_1$. In view of (5.8), we arrive at the inequality of Theorem 5.1. $\square$

Acknowledgement. We are grateful to V.G. Mazya for the reference to the work of I.S. Kac and M.G. Krein.

Research of the first author is supported in part by the NSF (grant no. DMS-0706866).

References

1. Artola, M.: Unpublished manuscript.
2. Bobkov, S.G., Götze, F.: Exponential integrability and transportation cost related to logarithmic Sobolev inequalities. J. Funct. Anal. **163**, no. 1, 1–28 (1999)
3. Birkhoff, G., Rota, G-C.: Ordinary Differential Equations. Xerox College Publ. (1969)
4. Chen, L.H.Y., Lou, J.-H.: A Characterization of Probability Measures which Admit Poincaré Inequalities. Preprint (1991)
5. Ince, E.L.: Ordinary Differential Equations. Dover Publ. (1956)
6. Jost, J., Li-Jost, X.: Calculus of Variations. Cambridge Univ. Press, Cambridge (1998)
7. Kac, I.S., Krein, M.G.: Criteria for the discreteness of the spectrum of a singular string (Russian). Izv. VUZov. Matematika, no. 2(3), 136–153 (1958)
8. Levin, A.Yu.: A comparison principle for second-order differential equations (Russian). Dokl. Akad. Nauk SSSR 783–786 (1960); English transl.: Sov. Math., Dokl. **1**, 1313–1316 (1960)

9. Mazya, V.G.: Sobolev Spaces. Springer-Verlag, Berlin - New York (1985)
10. Miclo, L.: An example of application of discrete Hardy's inequalities. Markov Process. Relat. Fields **5**, no. 3, 319–330, (1999)
11. Muckenhoupt, B.: Hardy's inequality with weights. Stud. Math. **44**, 31–38 (1972)
12. Reid, W.T.: Ordinary Differential Equations. John Wiley & Sons (1971)
13. Ross, S.L.: Differential Equations. Xerox (1974)
14. Talenti, G.: Osservazioni sopra una classe di disuguaglianze. Rend. Sem. Mat. Fis. Milano **39**, 171–185 (1969)
15. Tomaselli, G.: A class of inequalities. Boll. Unione Mat. Ital. **21**, 627–631 (1969)