# Distributions with Slow Tails and Ergodicity of Markov Semigroups in Infinite Dimensions

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**Abstract** We discuss some geometric and analytic properties of probability distributions that are related to the concept of weak Poincaré type inequalities. We deal with isoperimetric and capacitary inequalities of Sobolev type and applications to finite-dimensional convex measures with weights and infinite-dimensional Gibbs measures. As one of the basic tools, V. G. Mazya's capacitary analogue of the co-area inequality is adapted to the setting of metric probability spaces.

## 1 Weak Forms of Poincaré Type Inequalities

In this paper, we discuss some geometric and analytic properties of probability distributions, such as embeddings, concentration, and convergence of the associated semigroups, that are related to the concept of weak Poincaré type inequalities. Such inequalities may have different forms and appear in different contexts and settings. We mainly restrict ourselves to the setting of an arbitrary metric probability space, say, $(M, d, \mu)$ (keeping in mind the Euclidean space $\mathbf{R}^n$ as a basic space and source for various examples). We will be focusing on the following definition.

**Definition.** We say that $(M, d, \mu)$ satisfies a *weak Poincaré type inequality* with rate function $C(p)$, $1 \leqslant p < 2$, if for any bounded, locally Lipschitz function $f$ on $M$ with $\mu$-mean zero

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$$\|f\|_p \ \leqslant \ C(p)\, \|\nabla f\|_2 \quad \forall\, p \in [1,2). \tag{1.1}$$

More precisely, (1.1) involves a parameter family of Poincaré type inequalities that are controlled by a certain parameter function. Here, we use the standard notation $\|f\|_p = \Big( \int |f|^p \, d\mu \Big)^{1/p}$ for the $L^p$-norm, as well as $\|\nabla f\|_2 = \Big( \int |\nabla f|^2 \, d\mu \Big)^{1/2}$. Note that it is a rather convenient way to understand the modulus of the gradient in general as the function

$$|\nabla f(x)| = \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x,y)}, \quad x \in M$$

(with the convention that $|\nabla f(x)| = 0$ if $x$ is an isolated point in $M$). By saying that $f$ is "locally Lipschitz" we mean that $f$ has a finite Lipschitz seminorm on every ball in $M$, so that $|\nabla f(x)|$ is everywhere finite. Once (1.1) holds for all bounded locally Lipschitz $f$, it continues to hold for all unbounded locally Lipschitz functions with $\mu$-mean zero, as long as the right-hand side of (1.1) is finite. (The latter implies the finiteness of $\|f\|_p$ for all $p < 2$.)

As a more general scheme, one could start from a probability space $(M, \mu)$, equipped with some (local or discrete) Dirichlet form $\mathcal{E}(f,f)$, and to consider the inequalities

$$\|f\|_p \ \leqslant \ C(p) \sqrt{\mathcal{E}(f,f)}, \quad 1 \leqslant p < 2, \tag{1.2}$$

within the domain $\mathcal{D}$ of the Dirichlet form. Within the metric probability space framework, we thus put $\mathcal{E}(f,f) = \|\nabla f\|_2^2$. But one may also study (1.2) in the setting of a finite graph or, more generally, of Markov kernels, or the setting of Gibbs measures.

The main idea behind (1.1)–(1.2) is to involve in analysis more probability distributions and to quantify their possible analytic and other properties by means of the rate function. Indeed, if $C(p)$ may be chosen to be a constant, we arrive at the usual form of the Poincaré type inequality

$$\lambda_1 \operatorname{Var}_\mu(f) \ \leqslant \ \mathcal{E}(f,f), \tag{1.3}$$

where

$$\operatorname{Var}_\mu(f) = \int f^2 \, d\mu - \Big( \int f \, d\mu \Big)^2$$

stands for the variance of $f$ under $\mu$. This inequality itself poses a rather strict constraint on the measure $\mu$. For example, under (1.3) in the setting of a metric probability space $(M, d, \mu)$, any Lipschitz function $f$ on $M$ must have a finite exponential moment. This property, discovered by Herbst [22] and later by Gromov and Milman [20] and by Borovkov and Utev [13], may be stated as a deviation inequality

$$\mu\{|f| > t\} \leqslant C\, e^{-c\sqrt{\lambda_1}\, t}, \quad t > 0, \tag{1.4}$$

with some positive absolute constants $C$ and $c$, where for normalization reasons it is supposed that $\|f\|_{\rm Lip} \leqslant 1$ and $\int f\, d\mu = 0$. (The best constant in the exponent is known to be $c = 2$ and it is attained for the one-sided exponential distribution on the real line $M = \mathbf{R}$, cf. [7].) With a proper understanding of the Lipschitz property, discrete and more general analogues of (1.4) also hold under (1.3) (cf., for example, [2, 1, 24, 25]).

Another classical line of applications of the usual Poincaré type inequality deals with the Markov semigroup $P_t$ of linear operators associated to $\mu$ on $\mathbf{R}^n$ (or other Riemannian manifold). This semigroup has a generator $L$, which may be introduced via the equality

$$\mathcal{E}(f,g) = -\int f\, Lg\, d\mu, \quad f,g \in \mathcal{D},$$

so that $P_t = e^{tL}$ in the operator sense. Under (1.3) and mild technical assumptions, every $P_t$ represents a contraction on $L^2(\mu)$, i.e., $P_t$ may be extended from $\mathcal{D}$ as a linear continuous operator acting on the whole space $L^2(\mu)$ with the operator norm $\|P_t\| \leqslant 1$. Moreover, for any $f \in L^2(\mu)$,

$$\mathrm{Var}_\mu(P_t f) \leqslant \mathrm{Var}_\mu(f)\, e^{-\lambda_1 t}, \quad t > 0, \tag{1.5}$$

which expresses the $L^2(\mu)$ exponential ergodicity property of the Markov semigroup.

The exponential bounds such as (1.4)–(1.5) do not hold any longer without the hypothesis on the presence of the usual Poincaré type inequality. However, one may hope to get weaker conclusions under weaker assumptions, such as the weak Poincaré type inequality (1.1). In the latter case, the rate of growth of $C(p)$ as $p \to 2$ turns out to be responsible for the strength of deviations of Lipschitz functions and for the rate of convergence of $P_t f$ to a constant function, as well.

As a result, in the general situation more freedom in choosing suitable rate function $C(p)$ will allow us to involve more interesting probability spaces, especially those without finite exponential moments. In this connection it should be noted that another kind of inequalities, which serve this aim, is described by the weak forms of Poincaré type inequalities, that involve an oscillation term $\mathrm{Osc}\,(f) = \mathrm{ess\,sup}\, f - \mathrm{ess\,inf}\, f$ with respect to $\mu$. Namely, one considers

$$\mathrm{Var}_\mu(f) \leqslant \beta(s)\, \mathcal{E}(f,f) + s\, \mathrm{Osc}\,(f)^2. \tag{1.6}$$

These inequalities are supposed to hold for all $s > 0$ with some function $\beta$, so that the case of the constant function $\beta(s) = 1/\lambda_1$ also returns us to the usual Poincaré inequalities.

The inequalities with a free parameter have a long history in analysis, including, for example, [32, 17, 16, 18, 27, 21, 5, 6] and many others. The weak Poincaré type inequalities (1.6) have recently been studied by Röckner and Wang [35, 37], as an approach to the problem on the slow rates in the convergence of the associated Markov semigroups in $\mathbf{R^n}$. This work was motivated by Liggett [27], who considered similar multiplicative forms of (1.6). In the setting of Riemannian manifolds, Barthe, Cattiaux, and Roberto [4] studied the weak Poincaré type inequalities from the point of view of concentration and connected them with the family of capacitary inequalities, a classical object in the theory of Sobolev spaces. Such inequalities go back to the pioneering works of V. G. Maz'ya in the 60s and 70s; let us only mention [28], his book [29], and a nice exposition given by A. Grigoryan in [19]. See also [15], where entropic versions of (1.6) are treated. On the other hand, although weak Poincaré type inequalities (1.1) should certainly be of independent alternative interest, they seemed to attract much less attention. And for several reasons one may wonder how to fill this gap.

We explore, how the weak forms of Poincaré type inequalities (1.1)/(1.3) and (1.5) are related to each other (Section 8). Note that for probability measures on the real line, all the forms may be reduced to Hardy type inequalities with weights and this way they may be characterized explicitly in terms of the density of a measure (cf. [31, 29]). One obvious advantage of (1.1) over (1.6) is that one may freely apply (1.1) to unbounded functions, while (1.6) is more delicate in this respect. In fact, the relation (1.1), taken as a potential "nice" hypothesis, gives rise to a larger family of Poincaré type inequalities between the norms of $f$ and $|\nabla f|$ in Lebesgue spaces. This property, which we briefly discuss in Section 2, is usually interpreted as kind of embedding theorems. It is illustrated in Section 3 in the problem of large deviations of Lipschitz functionals. Sections 4 and 5 are technical, with aim to create tools to estimate the rate function for classes of measures on the Euclidean space under certain convexity conditions (cf. Sections 6 and 7). In Section 10, we discuss consequences of our weak Poincaré inequalities for the $L^p$ decay to equilibrium of Markov semigroups in $\mathbf{R^n}$. But before, in Section 9, we introduce the notation and recall classical arguments, that are used in the presence of the usual Poincaré inequalities.

Later we extend the corresponding idea to infinite dimensional situation in Section 11, where, in particular, we prove a stretched exponential decay for a product case. As we demonstrate there, it is the infinite dimensional case in which our more general than (1.6) inequalities play an important role in estimates of the decay rates. Finally, in the last section, we prove a weak Poincaré inequality for Gibbs measures with slowly decaying tails in the region of strong mixing. Using this result, we obtain an estimate for the decay to equilibrium in $L^2$ for all Lipschitz cylinder functions with the same stretched exponential rate.

## 2 $L^p$-Embeddings under Weak Poincaré

Let us start with an abstract metric probability space $(M, d, \mu)$ satisfying the weak Poincaré type inequality

$$\|f - \mathbf{E}f\|_p \ \leqslant \ C(p)\, \|\nabla f\|_2, \quad 1 \leqslant p < 2, \tag{2.1}$$

with a (finite) rate function $C(p)$. For definiteness, we may put $C(p) = C(1)$ for $0 < p < 1$, although in some places we will consider (2.1) for all $p \in (0, 2)$ with rate function defined in $0 < p < 1$ in a different way. Here and in the sequel, we use the standard notation $\mathbf{E}f = \mathbf{E}_\mu f = \int f\, d\mu$ for the expectation of $f$ under the measure $\mu$.

Let $W^q(\mu)$ denote the space of all locally Lipschitz functions $g$ on $M$, equipped with the norm

$$\|f\|_{W^q} = \|\nabla f\|_q + \|f\|_1.$$

Clearly, the norm is getting stronger with the growing parameter $q$. From (2.1) it follows that $\|f\|_p \leqslant (1 + C(p))\, \|f\|_{W^2}$, which means that all $L^p(\mu)$, $1 \leqslant p < 2$, are embedded in $W^2(\mu)$. Therefore, one may wonder whether this property may be sharpened by replacing $W^2(\mu)$ with other spaces $W^q(\mu)$. The answer is affirmative and is given by the following assertion.

**Theorem 2.1.** *Given* $1 \leqslant p < q \leqslant +\infty$, $q \geqslant 2$, *for any locally Lipschitz* $f$ *on* $M$

$$\|f - \mathbf{E}f\|_p \ \leqslant \ C(p, q)\, \|\nabla f\|_q, \tag{2.2}$$

*with constants* $C(p, q) = \frac{12\, C(r)}{r}\, p$, *where* $\frac{1}{r} = \frac{1}{2} + \frac{1}{p} - \frac{1}{q}$.

Thus, $L^p(\mu)$ may be embedded in $W^q(\mu)$, whenever $1 \leqslant p < q \leqslant +\infty$ and $q \geqslant 2$.

In particular, $\frac{1}{r} = 1 - \frac{1}{q}$ for $p = 2$, so $r$ represents the dual exponent $q^* = \frac{q}{q-1}$ and $C(2, q) = \frac{24\, C(q^*)}{q^*} \leqslant 24\, C(q^*)$. Hence we obtain a dual variant of (2.1).

**Corollary 2.2.** *Under* (2.1), *for any bounded, locally Lipschitz function* $f$ *on* $M$

$$\|f - \mathbf{E}f\|_2 \ \leqslant \ 24\, C(q^*)\, \|\nabla f\|_q, \quad q > 2. \tag{2.3}$$

Now, let us turn to the proofs, which actually contain standard arguments. In the sequel, we will use the following elementary:

**Lemma 2.3.** *For any measurable function $f$ on a probability space $(M, \mu)$ with a median $m$ and for any $p \geqslant 1$*

$$\|f - m\|_p \ \leqslant\ 3 \inf_{c \in \mathbf{R}} \|f - c\|_p \,.$$

*Proof.* One may assume that the norm $\|f\|_p$ is finite and non-zero. Note that, in general, the median is not determined uniquely. Nevertheless, by the monotonicity of this multi-valued functional, for any median $m = m(f)$ of $f$ there is a median $m(|f|)$ of $|f|$ such that $|m(f)| \leqslant m(|f|)$. On the other hand, by the Chebyshev inequality,

$$\mu\{|f| > t\} \leqslant \frac{\|f\|_p^p}{t^p} < \frac{1}{2}\,,$$

as long as $t > 2^{1/p}\, \|f\|_p$, so $m(|f|) \leqslant 2^{1/p}\, \|f\|_p$ for any median of $|f|$. The two bounds yield

$$\|f - m(f)\|_p \leqslant \|f\|_p + |m(f)| \leqslant (1 + 2^{1/p})\, \|f\|_p \leqslant 3\, \|f\|_p.$$

Applying this to $f - c$ and noting that $m(f) - c$ is one of the medians of $f - c$, we arrive at the desired conclusion. □

Lemma 2.3 allows us to freely interchange medians and expectations in the weak Poincaré type inequality. This can be stated as follows.

**Lemma 2.4.** *Under the hypothesis* (2.1), *for any locally Lipschitz function $f$ on $M$ with median $m(f)$*

$$\|f - m(f)\|_p \ \leqslant\ C'(p)\, \|\nabla f\|_2, \quad 1 \leqslant p < 2, \tag{2.4}$$

*where $C'(p) = 3C(p)$. In turn,* (2.4) *implies* (2.1) *with $C(p) = 2C'(p)$.*

Indeed, by Lemma 2.3, $\|f - m(f)\|_p \leqslant 3\, \|f - \mathbf{E}f\|_p$, and thus (2.1) implies (2.4). On the other hand, assuming that $m(f) = 0$ and starting from (2.4), we get

$$\|f - \mathbf{E}f\|_p \leqslant \|f\|_p + |\mathbf{E}f| \leqslant 2\, \|f\|_p = 2\, \|f - m(f)\|_p \leqslant 2C'(p)\, \|\nabla f\|_2.$$

**Lemma 2.5.** *Assume that the metric probability space $(M, d, \mu)$ satisfies*

$$\|f\|_p \ \leqslant\ A(p)\, \|\nabla f\|_2, \quad 0 < p < 2, \tag{2.5}$$

*in the class of all locally Lipschitz functions $f$ on $M$ with median $m(f) = 0$. Then in the same class,*

$$\|f\|_p \leqslant 2A(p,q)\,\|\nabla f\|_q, \quad 0 < p < q \leqslant +\infty,\ q \geqslant 2, \tag{2.6}$$

*with constants* $A(p,q) = \frac{A(r)}{r}\,p$, *where* $\frac{1}{r} = \frac{1}{2} + \frac{1}{p} - \frac{1}{q}$.

Clearly, $A(p,2) = A(p)$, so that (2.6) generalizes (2.5) within a factor of 2. If (2.5) is only given for the range $1 \leqslant p < 2$, one may just put $A(p) = A(1)$ for $0 < p < 1$.

Note that, due to the assumption $p < q$, we always have $0 < r < 2$. The assumption $q \geqslant 2$ guarantees that $r \leqslant p$.

*Proof of Lemma 2.5.* We may assume that $2 < q < +\infty$ and $\|f\|_q < +\infty$. First, let $f \geqslant 0$ and $m(f) = 0$. Hence $\mu\{f = 0\} \geqslant \frac{1}{2}$. By the hypothesis (2.5), for any $r \in (0,2)$

$$\mathbf{E} f^r \leqslant A(r)^r\,(\mathbf{E}\,|\nabla f|^2)^{r/2}.$$

Apply this inequality to the function $f^{p/r}$, which is nonnegative and has median zero, as well as $f$. Then, using the Hölder inequality with exponents $\alpha, \beta > 1$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we get

$$\begin{aligned}\mathbf{E} f^p = \mathbf{E}\,(f^{p/r})^r &\leqslant A(r)^r\,\Big(\frac{p}{r}\Big)^r \Big(\mathbf{E}\,f^{2(\frac{p}{r}-1)}\,|\nabla f|^2\Big)^{r/2} \\ &\leqslant A(r)^r\,\Big(\frac{p}{r}\Big)^r \Big(\mathbf{E}\,f^{2\alpha\,(\frac{p}{r}-1)}\Big)^{r/2\alpha} \Big(\mathbf{E}\,|\nabla f|^{2\beta}\Big)^{r/2\beta}.\end{aligned} \tag{2.7}$$

Now let $\frac{1}{r} = \frac{1}{2} + \frac{1}{p} - \frac{1}{q}$ and choose $\alpha$ so that $2\alpha\,(\frac{p}{r} - 1) = p$, i.e., $\frac{1}{2\alpha} = \frac{1}{r} - \frac{1}{p}$. Since $q > 2$, we have $r < p$, so $\alpha > 0$. Moreover, $\alpha > 1 \Leftrightarrow \frac{1}{r} < \frac{1}{2} + \frac{1}{p}$ which is fulfilled. Also, put $\frac{1}{2\beta} = \frac{1}{q}$, so that $\beta = \frac{q}{2} > 1$. Then (2.7) turns into

$$\mathbf{E} f^p \leqslant A(r)^r\,\Big(\frac{p}{r}\Big)^r (\mathbf{E}\,f^p)^{r/2\alpha}\,(\mathbf{E}\,|\nabla f|^q)^{r/2\beta},$$

which is equivalent to

$$\|f\|_p \leqslant A(p,q)\,\|\nabla f\|_q. \tag{2.8}$$

In the general case, we split $f = f^+ - f^-$ with $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. Without loss of generality, let $|\nabla f(x)| = 0$, when $f(x) = 0$ (otherwise, we may work with functions of the form $T(f)$ with smooth $T$, approximating the identity function and satisfying $T'(0) = 0$). Then both $f^+$ and $f^-$ are nonnegative, have median at zero, and $|\nabla f^+| = |\nabla f|\,1_{\{f>0\}}$, $|\nabla f^-| = |\nabla f|\,1_{\{f<0\}}$. Hence, by the previous step (2.8) applied to these functions,

$$\int_{\{f>0\}} |f|^p\,d\mu \leqslant A(p,q)^p \bigg(\int_{\{f>0\}} |\nabla f|^q\,d\mu\bigg)^{p/q},$$

$$\int_{\{f<0\}} |f|^p \, d\mu \leqslant A(p,q)^p \left( \int_{\{f<0\}} |\nabla f|^q \, d\mu \right)^{p/q}.$$

Finally, adding these inequalities and using an elementary bound $a^s + b^s \leqslant 2(a+b)^s$ $(a, b \geqslant 0,\ 0 \leqslant s \leqslant 1)$, we arrive at the desired estimate (2.6). □

*Proof of Theorem 2.1.* By Lemmas 2.4 and 2.5, for any locally Lipschitz $f$ on $M$ with median $m(f)$, whenever $1 \leqslant p < q \leqslant +\infty$ and $q \geqslant 2$, we have

$$\|f - m(f)\|_p \leqslant \frac{6p\, C(r)}{r} \|\nabla f\|_q.$$

Another application of Lemma 2.4 doubles the constant on the right-hand side. □

## 3 Growth of Moments and Large Deviations

As another immediate consequence of Theorem 2.1, we consider the case $q = +\infty$. Then $\frac{1}{r} = \frac{p+2}{2p}$, and we obtain the following assertion.

**Corollary 3.1.** *Under the weak Poincaré type inequality* (2.1)*, any Lipschitz function $f$ on $M$ has finite $L^p$-norms and, if $\|f\|_{\mathrm{Lip}} \leqslant 1$ and $\mathbf{E}f = 0$,*

$$\|f\|_p \leqslant 6(p+2)\, C\left(\frac{2p}{p+2}\right), \quad p \geqslant 1. \tag{3.1}$$

In the case of the usual Poincaré type inequality,

$$\lambda_1 \operatorname{Var}_\mu(f) \leqslant \int |\nabla f|^2 \, d\mu,$$

we have $C(p) = \frac{1}{\sqrt{\lambda_1}}$, and the inequality (3.1) gives

$$\|f\|_p \leqslant \frac{6(p+2)}{\sqrt{\lambda_1}}, \quad p \geqslant 1.$$

Up to a universal constant $c > 0$, the latter may also be stated as a large deviation bound $\mu\{\sqrt{\lambda_1}\, |f| \geqslant t\} \leqslant 2\, e^{-ct}$ $(t \geqslant 0)$ or, equivalently, as

$$\|f\|_{\psi_1} \leqslant \frac{1}{c\sqrt{\lambda_1}} \tag{3.2}$$

in terms of the Orlicz norm generated by the Young function $\psi_1(t) = e^{|t|} - 1$.

Thus, Theorem 2.1 may be viewed as a generalization of the Gromov–Milman theorem [20] on the concentration in the presence of the Poincaré type inequality. Let us give one more specific example by imposing the condition

$$C(p) \leqslant \frac{a}{(2-p)^\gamma}, \quad 1 \leqslant p < 2, \tag{3.3}$$

with some parameters $a, \gamma \geqslant 0$. In particular, if $\gamma = 0$, we return to the usual Poincaré type inequality.

**Corollary 3.2.** *Let $(M,d,\mu)$ satisfy the weak Poincaré type inequality* (2.1) *with a rate function admitting the polynomial growth* (3.3)*. For any function $f$ on $M$ with $\|f\|_{\rm Lip} \leqslant 1$ and $\mathbf{E} f = 0$*

$$\mu\{|f| \geqslant t\} \leqslant 2^{\gamma+1} \exp\Big\{-c_1(\gamma+1)\Big(\frac{c_2 t}{a}\Big)^{1/(\gamma+1)}\Big\}, \quad t > 0, \tag{3.4}$$

*where $c_1$ and $c_2$ are positive numerical constants.*

*Proof.* According to Corollary 3.1, for any $p \geqslant 1$

$$\|f\|_p \leqslant 6\,(p+2)\,\frac{a}{(2-\frac{2p}{p+2})^\gamma} = 6a \cdot 4^{-\gamma}(p+2)^{\gamma+1} \leqslant 18\,a \cdot (3/4)^\gamma p^{\gamma+1},$$

where we used $p + 2 \leqslant 3p$ at the last step. Hence, in the range $p \geqslant 1$, we have got the bound $\mathbf{E}\,|f|^p \leqslant (Cp)^{p(\gamma+1)}$ with a constant given by $C^{\gamma+1} = 18 \cdot (3/4)^\gamma\, a$. This bound may a little be weakened as

$$\mathbf{E}\,|f|^p \leqslant 2^{\gamma+1}(Cp)^{p(\gamma+1)} \tag{3.5}$$

to serve also the values $0 < p < 1$. Indeed, then we may use $\|f\|_p \leqslant \|f\|_1 \leqslant C^{\gamma+1}$, so $\mathbf{E}\,|f|^p \leqslant C^{p(\gamma+1)}$. Hence (3.5) would follow from $1 \leqslant 2p^p$, which is true since, on the positive half-axis, the function $2p^p$ is minimized at $p = \frac{1}{e}$ and has the minimum value $2e^{-1/e} > 1$.

Thus, (3.5) holds in the range $p > 0$. Now, by the Chebyshev inequality, for any $t > 0$

$$\mu\{|f| \geqslant t\} \leqslant \frac{\mathbf{E}\,|f|^p}{t^p} \leqslant 2^{\gamma+1}\,\frac{(Cp)^{p(\gamma+1)}}{t^p} = 2^{\gamma+1}\,(Dq)^q,$$

where $q = p\,(\gamma+1)$ and $D = \frac{C}{(\gamma+1)\,t^{1/(\gamma+1)}}$. The quantity $(Dq)^q$ is minimized, when $q = 1/(De)$, and the minimum is

$$e^{-1/(De)} = \exp\Big\{-\frac{(\gamma+1)\,t^{1/(\gamma+1)}}{Ce}\Big\} = \exp\Big\{-\frac{4\,(\gamma+1)\,t^{1/(\gamma+1)}}{3e\,(24\,a)^{1/(\gamma+1)}}\Big\}.$$

Thus, we arrive at (3.4) with $c_1 = 4/(3e)$ and $c_2 = 1/24$. □

In analogue with the usual Poincaré type inequality and similarly (3.2), the deviation inequality (3.4) of Corollary 3.2 may be restated equivalently in terms of the Orlicz norm generated by the Young function

$$\psi_{1/(\gamma+1)}(t) = \exp\{|t|^{1/(\gamma+1)}\} - 1.$$

Indeed, arguing in one direction, we consider $\xi = |f|^{1/(\gamma+1)}$ as a random variable on $(\mathbf{R}^n, \mu)$ and write (3.4) as

$$\mu\{\xi > t\} \leqslant Ae^{-Bt}, \quad t > 0,$$

with parameters $A = 2^{\gamma+1}$, $B = c_1(\gamma+1)\,(\frac{c_2}{a})^{1/(\gamma+1)}$. Then for any $r \in (0, B)$

$$\mathbf{E}\, e^{r\xi} - 1 = r\int_0^{+\infty} e^{rt}\mu\{\xi > t\}\, dt \leqslant Ar\int_0^{+\infty} e^{-(B-r)t}\, dt = \frac{Ar}{B-r} = 1$$

if $r = r_0 = \frac{B}{A+1}$. Hence $\mathbf{E}\,\exp\{r_0|f|^{1/(\gamma+1)}\} \leqslant 2$, which means that

$$\|f\|_{\psi_{1/(\gamma+1)}} \leqslant \frac{1}{r_0^{\gamma+1}} = \frac{(A+1)^{\gamma+1}}{B^{\gamma+1}} = \frac{a}{c_2}\,\frac{(2^{\gamma+1}+1)^{\gamma+1}}{(c_1(\gamma+1))^{\gamma+1}}.$$

Thus, under (3.3), up to some constant $c_\gamma$ depending on $\gamma$ only, we get

$$\|f\|_{\psi_{1/(\gamma+1)}} \leqslant c_\gamma a.$$

## 4 Relations for $L^p$-Like Pseudonorms

To give some examples of metric probability spaces satisfying weak Poincaré type inequalities, we need certain relations for $L^p$-like pseudonorms, which we discuss in this section. For a measurable function $f$ on the probability space $(M, \mu)$ and $q, r > 0$ we introduce the following standard notation. Put

$$\|f\|_q = \Big(\int |f|^q\, d\mu\Big)^{1/q}$$

and

$$\|f\|_{r,1} = \int_0^{+\infty} \mu\{|f| > t\}^{1/r}\, dt, \quad \|f\|_{r,\infty} = \sup_{t>0}\Big[t\,\mu\{|f| > t\}^{1/r}\Big].$$

As for how these quantities are related, there is the following elementary (and apparently well-known) statement: If $0 < q < r$, then

$$\|f\|_{r,1} \geqslant \|f\|_{r,\infty} \geqslant \Big(\frac{r-q}{r}\Big)^{1/q} \|f\|_q.$$

In particular,

$$\|f\|_{r,1} \geqslant \left(\frac{r-q}{r}\right)^{1/q} \|f\|_q. \tag{4.1}$$

However, the constant on the right-hand side is not optimal and may be improved, when $q$ and $r$ approach 1.

**Lemma 4.1.** *If* $0 < q < r \leqslant 1$, *then*

$$\|f\|_{r,1} \geqslant \left(\frac{r-q}{r}\right)^{1/q-1} \|f\|_q. \tag{4.2}$$

To see the difference between (4.1) and (4.2), we note that $\|f\|_{r,1} = \|f\|_1$ for the value $r = 1$ and, letting $q \to 1^-$, we obtain equality in (4.2), but not in (4.1).

*Proof.* Introduce the distribution function $F(t) = \mu\{|f| \leqslant t\}$ and put $u(t) = 1 - F(t)$. Since $u \leqslant 1$, for any $t > 0$

$$\|f\|_q^q = \int_0^{+\infty} s^q \, dF(s) = \int_0^t u(s)\, ds^q + \int_t^{+\infty} u(s)\, ds^q \leqslant t^q + q \int_t^{+\infty} s^{q-1}\, u(s)\, ds.$$

Let $0 < r < 1$. By the Hölder inequality with exponents $p = \frac{1}{r}$ and $p^* = \frac{p}{p-1}$, we have

$$\begin{aligned} \int_t^{+\infty} s^{q-1}\, u(s)\, ds &\leqslant \|s^{q-1}\|_{L^{p^*}(t,+\infty)}\, \|u(s)\|_{L^p(t,+\infty)} \\ &= \left(\int_t^{+\infty} s^{p^*(q-1)}\, ds\right)^{1/p^*} \left(\int_t^{+\infty} u(s)^p\, ds\right)^{1/p}. \end{aligned}$$

The last integral may be bounded from above just by $\|f\|_{r,1} = \displaystyle\int_0^{+\infty} u(s)^p\, ds$. Note that $p^*(q-1) < -1$; moreover,

$$\begin{aligned} p^*(q-1) + 1 &= -\frac{1-pq}{p-1} = -\frac{r-q}{1-r}, \\ \frac{p^*(q-1)+1}{p^*} &= -\frac{1-pq}{p-1}\, \frac{p-1}{p} = -(r-q). \end{aligned}$$

Hence the pre-last integral is convergent and

$$\left(\int_t^{+\infty} s^{p^*(q-1)}\, ds\right)^{1/p^*} = \frac{t^{\frac{p^*(q-1)+1}{p^*}}}{(-p^*(q-1)-1)^{1/p^*}} = \left(\frac{1-r}{r-q}\right)^{1-r} t^{-(r-q)}$$

since $\frac{1}{p^*} = \frac{p-1}{p} = 1 - r$. Thus,

$$\|f\|_q^q \leqslant t^q + q\left(\frac{1-r}{r-q}\right)^{1-r} t^{-(r-q)} \, \|f\|_{r,1}^r.$$

It remains to optimize over all $t > 0$ on the right-hand side. Changing the variable $t^q = s$, we write

$$\|f\|_q^q \leqslant \varphi(s) \equiv s + \frac{C}{\alpha} \, s^{-\alpha},$$

where $\alpha = \frac{r}{q} - 1$ and

$$C = q\left(\frac{1-r}{r-q}\right)^{1-r} \left(\frac{r}{q} - 1\right) \|f\|_{r,1}^r = (1-r)^{1-r}(r-q)^r \, \|f\|_{r,1}^r.$$

Since $\alpha > 0$, the function $\varphi$ is minimized at $s_0 = C^{1/(\alpha+1)}$ and, at this point,

$$\varphi(s_0) = C^{1/(\alpha+1)} + \frac{C}{\alpha} \, C^{-\alpha/(\alpha+1)} = \left(1 + \frac{1}{\alpha}\right) C^{1/(\alpha+1)}.$$

Note that $\alpha + 1 = \frac{r}{q}$ and $\frac{\alpha+1}{\alpha} = \frac{r}{r-q}$, so

$$C^{1/(\alpha+1)} = \left[(1-r)^{1-r}(r-q)^r \, \|f\|_{r,1}^r\right]^{q/r} = (1-r)^{q(\frac{1}{r}-1)} \, (r-q)^q \, \|f\|_{r,1}^q$$

and

$$\varphi(s_0) = \frac{r}{r-q} \, (1-r)^{q(\frac{1}{r}-1)} \, (r-q)^q \, \|f\|_{r,1}^q.$$

Therefore,

$$\|f\|_q \leqslant \varphi(s_0)^{1/q} = \left(\frac{r}{r-q}\right)^{\frac{1}{q}-1} r(1-r)^{\frac{1}{r}-1} \, \|f\|_{r,1}.$$

It remains to note that $r(1-r)^{\frac{1}{r}-1} \leqslant 1$, whenever $0 < r < 1$. □

## 5 Isoperimetric and Capacitary Conditions

Here, we focus on general necessary and sufficient conditions for weak Poincaré type inequalities to hold on a metric probability space $(M, d, \mu)$. Sufficient conditions are usually expressed in terms of the isoperimetric function of the measure $\mu$, so it is natural to explore the role of isoperimetric inequalities. By an *isoperimetric inequality* one means any relation

$$\mu^+(A) \geqslant I(\mu(A)), \quad A \subset M, \tag{5.1}$$

connecting the outer Minkowski content or $\mu$-perimeter

$$\mu^+(A) = \liminf_{\varepsilon \to 0^+} \frac{\mu(A^\varepsilon) - \mu(A)}{\varepsilon}$$
$$= \liminf_{\varepsilon \to 0^+} \frac{\mu\{x \in M \setminus A : \exists a \in A,\ d(x,a) < \varepsilon\}}{\varepsilon}$$

with $\mu$-size in the class of all Borel sets $A$ in $M$ with measure $0 < \mu(A) < 1$. (Here $A^\varepsilon$ denotes an open $\varepsilon$-neighborhood of $A$.)

The function $I$, appearing in (5.1), may be an arbitrary nonnegative function, defined on the unit interval (0,1). If this function is optimal, it is often referred to as the isoperimetric function or the isoperimetric profile of the measure $\mu$.

To any nonnegative function $I$ on $(0, \frac{1}{2}]$ we associate a nondecreasing function $C_I(r)$ given by

$$\frac{1}{C_I(r)} = \inf_{0 < t \leqslant \frac{1}{2}} \Big[ I(t)\, t^{-1/r} \Big], \quad 0 < r < 1. \tag{5.2}$$

One of our aims is to derive the following assertion.

**Theorem 5.1.** *In the presence of the isoperimetric inequality* (5.1)*, the space* $(M, d, \mu)$ *satisfies the weak Poincaré type inequality*

$$\|f - \mathbf{E} f\|_p \leqslant C(p)\, \|\nabla f\|_2, \quad 0 < p < 2,$$

*with rate function*

$$C(p) = 8 \inf_{\frac{2p}{2+p} < r < 1} \bigg[ C_I(r) \Big( \frac{r}{r - \frac{2p}{2+p}} \Big)^{\frac{2-p}{2p}} \bigg]. \tag{5.3}$$

We first consider one important particular case.

**Lemma 5.2.** *Given* $c > 0$ *and* $0 < r \leqslant 1$*, we assume that* $(M, d, \mu)$ *satisfies*

$$\mu^+(A) \geqslant c\, \mu(A)^{1/r} \tag{5.4}$$

*for all Borel sets* $A \subset M$ *with* $0 < \mu(A) \leqslant \frac{1}{2}$*. Then for any locally Lipschitz function* $f \geqslant 0$ *on* $M$ *with median zero and for all* $q \in (0, r)$

$$\|f\|_q \leqslant \frac{1}{c} \Big( \frac{r}{r - q} \Big)^{1/q - 1} \int |\nabla f|\, d\mu. \tag{5.5}$$

For the proof, we recall the well-known co-area formula which remains to hold in the form of an inequality for arbitrary metric probability spaces (cf. [10]). Namely, for any function $f$ on $M$ having a finite Lipschitz seminorm

$$\int |\nabla f| \, d\mu \geqslant \int_{-\infty}^{+\infty} \mu^+\{f > t\} \, dt.$$

Note that the function $t \to \mu^+\{f > t\}$ is always Borel measurable for continuous $f$, so the second integral makes sense. Hence, by Lemma 4.1, if $\|f\|_{\mathrm{Lip}} < +\infty$,

$$\int |\nabla f| \, d\mu \geqslant c \int_0^{+\infty} \mu\{f > t\}^{1/r} \, dt = c \, \|f\|_{r,1} \geqslant c \left(\frac{r-q}{r}\right)^{1/q-1} \|f\|_q.$$

A simple truncation argument extends this inequality to all locally Lipschitz $f \geqslant 0$.

*Proof of Theorem 5.1.* By the definition (5.2), whenever $0 < r < 1$, the space $(M, d, \mu)$ satisfies the isoperimetric inequality (5.4) with $c = 1/C_I(r)$, so the functional inequality (5.5) holds.

Let $f \geqslant 0$ be locally Lipschitz on $M$ with median zero. Given $0 < q < r < 1$, apply (5.5) to $f^{p/q}$ with $p > 0$ to be specified later on. Then

$$\begin{aligned}
\int f^p \, d\mu = \int (f^{p/q})^q \, d\mu &\leqslant \frac{1}{c^q} \left(\frac{r}{r-q}\right)^{1-q} \left(\int |\nabla f^{p/q}| \, d\mu\right)^q \\
&= \frac{1}{c^q} \left(\frac{r}{r-q}\right)^{1-q} \left(\frac{p}{q}\right)^q \left(\int f^{\frac{p}{q}-1} |\nabla f| \, d\mu\right)^q \\
&\leqslant \frac{1}{c^q} \left(\frac{r}{r-q}\right)^{1-q} \left(\frac{p}{q}\right)^q \left(\int f^{2(\frac{p}{q}-1)} \, d\mu\right)^{q/2} \left(\int |\nabla f|^2 \, d\mu\right)^{q/2},
\end{aligned}$$

where we used the Cauchy inequality at the last step. Choose $p$ so that $2\,(\frac{p}{q} - 1) = p$, i.e., $p = 2q/(2-q)$ or $q = 2p/(2+p)$. Then the obtained bound becomes

$$\left(\int f^p \, d\mu\right)^{1-q/2} \leqslant \frac{1}{c^q} \left(\frac{r}{r-q}\right)^{1-q} \left(\frac{2}{2-q}\right)^q \left(\int |\nabla f|^2 \, d\mu\right)^{q/2},$$

and, using $\frac{1-q/2}{q} = \frac{1}{p}$ and $\frac{2}{2-q} < 2$, we get

$$\left(\int f^p \, d\mu\right)^{1/p} \leqslant \frac{2}{c} \left(\frac{r}{r-q}\right)^{1/q-1} \left(\int |\nabla f|^2 \, d\mu\right)^{1/2}.$$

By doubling the expression on the right-hand side like in the proof of Lemma 2.5, we may remove the condition $f \geqslant 0$ and thus get in the general locally Lipschitz case

$$\|f - m(f)\|_p \leqslant \frac{4}{c} \left(\frac{r}{r-q}\right)^{1/q-1} \|\nabla f\|_2$$

with $q = \frac{2p}{2+p}$, where $m(f)$ is a median of $f$ under $\mu$. Note that $q < 1 \Leftrightarrow p < 2$. Finally, by Lemma 2.4,

$$\|f - \mathbf{E}_\mu f\|_p \leqslant \frac{8}{c}\left(\frac{r}{r-q}\right)^{1/q-1} \|\nabla f\|_2.$$

It remains to take the infimum over all $r \in (q,1)$, and we arrive at the desired Poincaré type inequality with rate function (5.4). □

Remark 5.1. In order to get a simple upper bound for the rate function

$$C(p) = 8 \inf_{q<r<1} \left[C_I(r)\left(\frac{r}{r-q}\right)^{\frac{1}{q}-1}\right], \quad \text{where} \quad q = \frac{2p}{2+p},$$

in many interesting cases, one may just take

$$r = \frac{1+q}{2} = \frac{3p+2}{2(p+2)},$$

for example. In this case,

$$\left(\frac{r}{r-q}\right)^{\frac{1}{q}-1} = \left(\frac{1+q}{1-q}\right)^{\frac{1}{q}-1} = (1+s)^{2/s} < e^2$$

for $s = 2q/(1-q)$. Hence we obtain the following assertion.

**Corollary 5.3.** *In the presence of the isoperimetric inequality* (5.1) *with the associated function* $C_I(r)$*, the space* $(M,d,\mu)$ *satisfies the weak Poincaré type inequality*

$$\|f - \mathbf{E} f\|_p \leqslant C(p)\, \|\nabla f\|_2, \quad 0 \leqslant p < 2,$$

*with rate function* $C(p) = 8e^2\, C_I(\frac{3p+2}{2(p+2)})$.

In particular, if $\mu$ satisfies a Cheeger type isoperimetric inequality $\mu^+(A) \geqslant c\,\mu(A)$ $(0 < \mu(A) \leqslant \frac{1}{2})$, then $C_I(r)$ is bounded by $1/c$, and Corollary 5.3 yields the usual Poincaré type inequality

$$\|f - \mathbf{E} f\|_2 \leqslant \frac{C}{c}\, \|\nabla f\|_2$$

with a universal constant $C$. Thus, Theorem 5.1 includes the Maz'ya–Cheeger theorem (up to a multiplicative factor).

Consider a more general class of isoperimetric inequalities.

**Corollary 5.4.** *Assume that the metric probability space $(M, d, \mu)$ satisfies, for some $\alpha \geqslant 0$ and $c > 0$, an isoperimetric inequality*

$$\mu^+(A) \geqslant c \, \frac{t}{\log^{1/\alpha}(\frac{4}{t})}, \quad t = \mu(A), \ 0 < t \leqslant \frac{1}{2}.$$

*Then for some universal constant $C$ it satisfies the weak Poincaré type inequality with rate function*

$$C(p) = \frac{C}{c} \left( \frac{3}{2-p} \right)^{1/\alpha}, \quad 1 \leqslant p < 2.$$

*Proof.* First we show that, given $p > 1$, for all $t \in (0,1)$

$$\frac{t}{\log^{1/\alpha}(\frac{4}{t})} \geqslant \frac{[\alpha e(p-1)]^{1/\alpha}}{4^{p-1}} \, t^p. \tag{5.6}$$

Indeed, for any $C > 0$, replacing $t = 4s$, we can write

$$C \, \frac{t}{\log^{1/\alpha}(\frac{4}{t})} \geqslant t^p \Longleftrightarrow s^{\alpha(p-1)} \log \frac{1}{s^{\alpha(p-1)}} \leqslant \alpha(p-1) \left( \frac{C}{4^{p-1}} \right)^{\alpha}.$$

But $\sup_{u>0} [u \log \frac{1}{u}] = \frac{1}{e}$, so we are reduced to $\frac{1}{e} \leqslant \alpha(p-1) \, (\frac{C}{4^{p-1}})^{\alpha}$, where the best constant is $C = \frac{4^{p-1}}{[\alpha e(p-1)]^{1/\alpha}}$.

Now, using the definition (5.2) with $r = 1/p$ and applying (5.6), we conclude that $(M, d, \mu)$ satisfies an isoperimetric inequality with the associated function

$$C_I(r) = \frac{C}{c} \, \frac{4^{\frac{1}{r}-1}}{(\frac{1}{r}-1)^{1/\alpha}}, \quad \text{where} \ \ C = \frac{1}{(\alpha e)^{1/\alpha}}.$$

Take $r = \frac{1+q}{2} = \frac{3p+2}{2(p+2)}$ with $1 \leqslant p < 2$ as in Corollary 5.3 ($q = \frac{2p}{2+p}$). Since $r \geqslant \frac{5}{6}$, we have $4^{\frac{1}{r}-1} \leqslant 4^{1/5}$. Also $\frac{1}{r} - 1 = \frac{1+q}{1-q} = \frac{2-p}{2+3p} \geqslant \frac{2-p}{8}$ and $\alpha^{1/\alpha} \geqslant e^{-e}$. Therefore,

$$C_I(r) \leqslant \frac{4^{1/5} \, e^e}{c} \left( \frac{8/e}{2-p} \right)^{1/\alpha}.$$

It remains to apply Corollary 5.3. □

Although the isoperimetric inequalities may serve as convenient sufficient conditions for the week Poincaré type inequalities, in general they are not necessary. To speak about both necessary and sufficient conditions expressed in terms of geometric characteristics of a measure $\mu$, one has to involve the concept of the capacity of sets, which is close to, but different than the concept of the $\mu$-perimeter.

Given a metric space $(M, d)$ with a Borel (positive) measure $\mu$ and a pair of sets $A \subset \Omega \subset M$ such that $A$ is closed and $\Omega$ is open in $M$, the relative $\mu$-capacity of $A$ with respect to $\Omega$ is defined as

$$\mathrm{cap}_\mu(A, \Omega) = \inf \int |\nabla f|^2 \, d\mu,$$

where the infimum is taken over all locally Lipschitz functions $f$ on $M$, such that $f \geqslant 1$ on $A$ and $f = 0$ outside $\Omega$. The capacity of the set $A$ is $\mathrm{cap}_\mu(A) = \inf_\Omega \mathrm{cap}_\mu(A, \Omega)$. This definition is usually applied, when $M$ is the Euclidean space $\mathbf{R}^n$ equipped with the Lebesgue measure $\mu$ (or for Riemannian manifolds, cf. [29, 19]). To make the definition workable in the setting of a metric probability space $(M, d, \mu)$, so that to efficiently relate it to the energy functional $\int \nabla f|^2 \, d\mu$, the relative capacity should be restricted to the cases such as $\mu(\Omega) \leqslant 1/2$.

Thus, let $(M, d, \mu)$ be a metric probability space and $A$ a closed set in $M$ of measure $\mu(A) \leqslant 1/2$. Following [4], we define the $\mu$-capacity of $A$ by

$$\mathrm{cap}_\mu(A) \; = \inf_{\mu(\Omega) \leqslant 1/2} \mathrm{cap}_\mu(A, \Omega) = \inf \left\{ \int |\nabla f|^2 \, d\mu : 1_A \leqslant f \leqslant 1_\Omega \right\}, \quad (5.7)$$

where the first infimum runs over all open sets $\Omega \subset M$ containing $A$ and with measure $\mu(\Omega) \leqslant 1/2$, and the second one is taken over all such $\Omega$'s and all locally Lipschitz functions $f : M \rightarrow [0, 1]$ such that $f = 1$ on $A$ and $f = 0$ outside $\Omega$.

Note that, by the regularity of measure, we have $\mu(A^\varepsilon) \downarrow \mu(A)$ as $\varepsilon \downarrow 0$. Hence, if $\mu(A) < 1/2$, open sets $\Omega$ such that $A \subset \Omega$, $\mu(\Omega) \leqslant 1/2$ do exist, so the second infimum is also well defined and the definition makes sense. If $\mu(A) = 1/2$ and $\Omega$ does not exist, let us agree that the capacity is undefined (actually, this case does not appear when dealing with functional inequalities).

With this definition the measure capacity inequalities on $(M, d, \mu)$ take the form

$$\mathrm{cap}_\mu(A) \geqslant J(\mu(A)), \quad (5.8)$$

where $J$ is a nonnegative function defined on $(0, \frac{1}{2}]$ and $A$ is any closed subset of $M$ with $\mu(A) \leqslant 1/2$, for which the capacity is defined.

To see, how (5.8) is related to the weak Poincaré type inequality

$$\|f - \mathbf{E}f\|_p \;\leqslant\; C(p) \, \|\nabla f\|_2, \quad 1 \leqslant p < 2, \quad (5.9)$$

we take a pair of sets $A \subset \Omega \subset M$ and a function $f$ as in the definition (5.7). Then $f$ has median zero under $\mu$ and, by Lemma 2.3,

$$\|f - \mathbf{E}f\|_p \geqslant \frac{1}{3} \, \|f\|_p \geqslant \frac{1}{3} \, (\mu(A))^{1/p}.$$

Therefore, by (5.9),

$$\int |\nabla f|^2 \, d\mu \geqslant \frac{1}{9C(p)^2} \, (\mu(A))^{1/p}.$$

Taking the infimum over all admissible $f$ and the supremum over all $p$, we get the following elementary assertion.

**Theorem 5.5.** *Under the Poincaré type inequality* (5.9), *the measure capacity inequality* (5.8) *holds with*

$$J(t) = \frac{1}{9} \sup_{1 \leqslant p < 2} \left[ \frac{t^{1/p}}{C(p)} \right]^2, \quad 0 < t \leqslant \frac{1}{2}.$$

In particular, the usual Poincaré type inequality, when $C(p) = 1/\sqrt{\lambda_1}$ is constant, implies that $\mathrm{cap}_\mu(A) \geqslant c\lambda_1 \mu(A)$ with a numerical constant $c > 0$ (cf. [4]).

To move in the opposite direction from (5.8) to (5.9), we need a capacitary analogue of the co-area formula or co-area inequality, which was used in the proof of Lemma 5.2. It has indeed been known since the works by Maz'ya [28, 29], and below we just adapt his result and the argument of [30] to the setting of a metric probability space.

**Lemma 5.6.** *For any locally Lipschitz function $f \geqslant 0$ on $M$ with $\mu$-dedian zero*

$$\int_{\{f>0\}} |\nabla f|^2 \, d\mu \geqslant \frac{1}{5} \int_0^{+\infty} \mathrm{cap}_\mu\{f \geqslant t\} \, dt^2. \tag{5.10}$$

Note that the capacity functional $A \to \mathrm{cap}_\mu(A)$ is nondecreasing, so the second integrand in (5.10) represents a nonincreasing function in $t > 0$. For a proof of (5.10), we consider (locally Lipschitz) functions of the form

$$g = \frac{1}{c_1 - c_0} \max\{\min\{f, c_1\} - c_0, 0\}, \quad \text{where} \ \ c_1 > c_0 > 0.$$

We have $g = 1$ on the closed set $A = \{f \geqslant c_1\}$ and $g = 0$ outside the open set $\Omega = \{f > c_0\}$. Since $\mu(\Omega) \leqslant 1/2$, by the definition of the capacity,

$$\int |\nabla g|^2 \, d\mu \geqslant \mathrm{cap}_\mu(A, \Omega) \geqslant \mathrm{cap}_\mu(A).$$

On the other hand, since the function $(c_1 - c_0)\, g$ represents a Lipschitz transform of $f$, we have $(c_1 - c_0)\, |\nabla g(x)| \leqslant |\nabla f(x)|$ for all $x \in M$. In addition, $g$ is constant on the open sets $\{f < c_0\}$ and $\{f > c_1\}$, so $|\nabla g| = 0$ on these sets. Therefore,

$$\int |\nabla g|^2 \, d\mu = \int_{\{c_0 \leqslant f \leqslant c_1\}} |\nabla g|^2 \, d\mu \leqslant \frac{1}{(c_1 - c_0)^2} \int_{\{c_0 \leqslant f \leqslant c_1\}} |\nabla f|^2 \, d\mu.$$

The two estimates yield

$$(c_1 - c_0)^2 \, \mathrm{cap}_\mu\{f \geqslant c_1\} \leqslant \int_{\{c_0 \leqslant f \leqslant c_1\}} |\nabla f|^2 \, d\mu$$

or, given $a \in (0,1)$, for any $t > 0$

$$\int_{\{at \leqslant f \leqslant t\}} |\nabla f|^2 \, d\mu \geqslant t^2 (1-a)^2 \, \mathrm{cap}_\mu\{f \geqslant t\}.$$

Now, we divide both sides by $t$ and integrate over $(0, +\infty)$. This leads to

$$\int_{\{f>0\}} |\nabla f|^2 \, d\mu \geqslant \frac{(1-a)^2}{\log(1/a)} \int_0^{+\infty} t \, \mathrm{cap}_\mu\{f \geqslant t\} \, dt.$$

The coefficient on the right-hand side is greater than $2/5$ for almost an optimal choice $a = 0.3$.

Now, we are prepared to derive from the capacitary inequality (5.8) a certain weak Poincaré type inequality. This may be done with arguments similar to the ones used in the proof of Lemma 5.2. To get an estimate of the rate function, consistent with what we have got in Theorem 5.5, let us assume that $(M, d, \mu)$ satisfies

$$\mathrm{cap}_\mu(A) \geqslant \sup_{0 \leqslant p < 2} \left[ \frac{\mu(A)^{1/p}}{C(p)} \right]^2, \quad 0 < \mu(A) \leqslant \frac{1}{2}, \tag{5.11}$$

with a given positive function $C(p)$ defined in $0 < p < 2$. Equivalently, we could start with the measure capacity inequality (5.8) with a “capacitary” function $J(t)$ and then (5.11) holds with

$$C_J(p) = \sup_{0 < t \leqslant 1/2} \frac{t^{1/p}}{\sqrt{J(t)}}, \quad 0 < p < 2. \tag{5.12}$$

Let $r = p/2$ and $q < r < 1$. Given a locally Lipschitz function $f \geqslant 0$ on $M$, we may combine Lemma 5.6 with Lemma 4.1, to get from (5.11) that

$$\int_{\{f>0\}} |\nabla f|^2 \, d\mu \geqslant c \int_0^{+\infty} \mu\{f \geqslant t\}^{1/r} \, dt^2 = c \int_0^{+\infty} \mu\{f^2 \geqslant t\}^{1/r} \, dt$$
$$= c \, \|f^2\|_{r,1} \geqslant c \left( \frac{r-q}{r} \right)^{1/q - 1} \|f^2\|_q,$$

where $c = \dfrac{1}{5C(p)^2} = \dfrac{1}{5C(2r)^2}$. Equivalently,

$$\|f\|_{2q}^2 \ \leqslant\ 5\,C(2r)^2\Big(\frac{r}{r-q}\Big)^{\frac{1-q}{q}} \int_{\{f>0\}} |\nabla f|^2\,d\mu. \tag{5.13}$$

If $f$ is not necessarily nonnegative, but has median zero, one may apply (5.13) to the functions $f^+$ and $f^-$, and, summing the corresponding inequalities, we will be led again to (5.13) for $f$. Moreover, by doubling the constant on the right, the assumption $m(f) = 0$ may be replaced with $\mathbf{E}f = 0$. Thus, in general,

$$\|f - \mathbf{E}f\|_{2q}^2 \ \leqslant\ 10\,C(2r)^2\Big(\frac{r}{r-q}\Big)^{\frac{1-q}{q}} \int |\nabla f|^2\,d\mu, \quad 0 < q < r < 1.$$

Finally, replacing $2q$ with the variable $p$, we arrive at the following assertion.

**Theorem 5.7.** *Under the hypothesis* (5.11)*, the weak Poincaré type inequality*

$$\|f - \mathbf{E}f\|_p \ \leqslant\ C'(p)\,\|\nabla f\|_2, \quad 0 < p < 2,$$

*holds with rate function*

$$C'(p) = \sqrt{10}\ \inf_{\frac{p}{2}<r<1}\bigg[C(2r)\Big(\frac{r}{r-p/2}\Big)^{\frac{1-p/2}{p}}\bigg]. \tag{5.14}$$

Alternatively, if we start with the measure capacity inequality (5.8) with a function $J(t)$, one may associate to it the function $C_J$ defined in (5.12), and then the rate function of the theorem will take the form

$$C'(p) = \sqrt{10}\ \inf_{\frac{p}{2}<r<1}\ \sup_{0<t\leqslant 1/2}\bigg[\frac{t^{1/(2r)}}{\sqrt{J(t)}}\Big(\frac{r}{r-p/2}\Big)^{\frac{1-p/2}{p}}\bigg].$$

In particular, like in Corollary 5.3, choosing in (5.14) the value $r = (1+q)/2$ with $q = p/2$ and using the bounds $(\frac{r}{r-q})^{\frac{1-q}{2q}} < e$ and $\sqrt{10}\,e < 9$ (just to simplify the numerical constant), one may take

$$C'(p) = 9\,C(1 + p/2) = 9 \sup_{0<t\leqslant 1/2}\bigg[\frac{t^{\frac{1}{1+p/2}}}{\sqrt{J(t)}}\bigg], \quad 0 < p < 2. \tag{5.15}$$

Thus, starting with the weak Poincaré type inequality (5.9) with rate function $C(p)$, we obtain a geometric (capacity) inequality of the form (5.11), which in turn leads to (5.9), however, with a somewhat worse rate function $C'(p)$. Nevertheless, in some interesting cases, these two rate functions are in essence equivalent as $p \to 2$. For example, as in Corollary 5.4, if $C(p) = C \cdot (2-p)^{-1/\alpha}$, then $C'(p) = 9 \cdot 2^{1/\alpha}\, C \cdot (2-p)^{-1/\alpha}$, which is of the same order. It is in this sense one may say that weak Poincaré type inequalities have an equivalent capacitary description.

## 6 Convex Measures

Here, we illustrate Theorem 5.1 and especially its Corollary 5.4 on the example of probability distributions on the Euclidean space $M = \mathbf{R}^n$ possessing certain convexity properties. The obtained results will be applied to the so-called convex measures introduced and studied in the works of Borell [11, 12].

A Borel probability measure $\mu$ is called *$\varkappa$-concave*, where $-\infty \leqslant \varkappa \leqslant 1$, if for all $t \in (0,1)$ it satisfies a Brunn–Minkowski type inequality

$$\mu(tA + (1-t)B) \;\geqslant\; [\, t\mu(A)^{\varkappa} + (1-t)\mu(B)^{\varkappa}\,]^{1/\varkappa} \tag{6.1}$$

in the class of all nonempty Borel sets $A, B \subset \mathbf{R}^n$.

When $\varkappa = 0$, the right-hand side of (6.1) is understood as $\mu(A)^t \mu(B)^{1-t}$ and then we arrive at the notion of a log-concave measure, previously considered by Prékopa [33, 34] and Leindler [26] (cf. also [14]). When $\varkappa = -\infty$, the right-hand side is understood as $\min\{\mu(A), \mu(B)\}$. The inequality (6.1) is getting stronger, as the parameter $\varkappa$ is increasing, so the case $\varkappa = -\infty$ describes the largest class, whose members are called *convex* or *hyperbolic probability measures.*

Borell gave a complete characterization of such measures. If $\mu$ is absolutely continuous with respect to the Lebesgue measure and is supported on some open convex set $K \subset \mathbf{R}^n$, the necessary and sufficient condition for $\mu$ to satisfy (6.1) is that it has a positive density $p$ on $K$ such that for all $t \in (0,1)$ and $x, y \in K$

$$p(tx + (1-t)y) \;\geqslant\; [\, tp(x)^{\varkappa_n} + (1-t)p(y)^{\varkappa_n}\,]^{1/\varkappa_n}\,, \tag{6.2}$$

where $\varkappa_n = \frac{\varkappa}{1-n\varkappa}$ (necessarily $\varkappa \leqslant \frac{1}{n}$). Thus, the $\varkappa$-concavity with $\varkappa < 0$ means that the density is representable in the form $p = V^{-\beta}$ for some positive convex function $V$ on $\mathbf{R}^n$, possibly taking an infinite value, where $\beta \geqslant n$ and $\varkappa = -\frac{1}{\beta - n}$.

Below we consider $\varkappa$-concave probability measures with $\varkappa < 0$. As was shown in [23] for the convex body case ($\varkappa = \frac{1}{n}$) and then in [8] for the general log-concave case ($\varkappa = 0$), any log-concave probability measure shares the usual Poincaré type inequality. This property fails when $\varkappa < 0$ even under strong integrability hypotheses. Nevertheless, with such additional hypotheses one may reach weak Poincaré type inequalities! More precisely, we will involve the condition that the distribution function $F(r) = \mu\{|x| \leqslant r\}$ of the

Euclidean norm has the tails $1 - F(r)$ decreasing to zero, as $r \to +\infty$, at worst as $e^{-ct^\alpha}$. As long as the parameter of the convexity $\varkappa$ is negative, there is no reason to distinguish between the case corresponding to the exponential tails with $\alpha \geqslant 1$ (which is typical for log-concave distributions) and the case of (relatively) heavy or slow tails, when $\alpha < 1$.

We need some preparations. Denote by $B_\rho$ an open Euclidean ball of radius $\rho > 0$ with center at the origin.

**Lemma 6.1.** *Any $\varkappa$-concave probability measure, $-\infty < \varkappa \leqslant 1$, satisfies the isoperimetric inequality*

$$2\rho\,\mu^+(A) \,\geqslant\, \frac{1 - [\,t^{1-\varkappa} + (1-t)^{1-\varkappa}]\,\mu(B_\rho)}{-\varkappa}, \tag{6.3}$$

*where $t = \mu(A)$, $0 < t < 1$, with arbitrary $\rho > 0$.*

In the log-concave case, the inequality (6.3) should read as

$$2\rho\,\mu^+(A) \,\geqslant\, t\log\frac{1}{t} + (1-t)\log\frac{1}{1-t} + \log\mu(B_\rho). \tag{6.4}$$

By the Prékopa–Leindler functional form of the Brunn–Minkowski inequality, (6.4) was derived in [8]. The arbitrary $\varkappa$-concave case was considered by Barthe [3], who applied an extension of the Prékopa–Leindler theorem in the form of Borell and Brascamp–Lieb. The inequality (6.3) was used in [3] to study the isoperimetric dimension of $\varkappa$-concave measures with $\varkappa > 0$. A direct proof of (6.3), not appealing to any functional form was given in [9].

To make the exposition self-contained, let us briefly remind the argument, which is based on the following representation for the $\mu$-perimeter, explicitly relating it to measure convexity properties. Namely, let a probability measure $\mu$ on $\mathbf{R}^n$ be absolutely continuous and have a continuous density $p(x)$ on an open supporting convex set, say $K$. It is easy to check that for any sufficiently "nice" set $A$, for example, a finite union of closed balls in $K$ or the complement in $\mathbf{R}^n$ to the finite union of such balls

$$\mu^+(A) = \lim_{\varepsilon\to 0^+} \frac{\mu((1-\varepsilon)A + \varepsilon B_\rho) + \mu((1-\varepsilon)\overline{A} + \varepsilon B_\rho) - 1}{2r\varepsilon}\,, \tag{6.5}$$

where $\overline{A} = \mathbf{R}^n \setminus A$. In the case of a $\varkappa$-concave $\mu$, it remains to apply the original convexity property (6.1) to the right-hand side of (6.5) to get

$$\mu^+(A) \geqslant \lim_{\varepsilon\to 0^+} \frac{((1-\varepsilon)\mu(A)^\varkappa + \varepsilon\mu(B_\rho)^\varkappa)^{1/\varkappa} + ((1-\varepsilon)\mu(\overline{A})^\varkappa + \varepsilon\mu(B_\rho)^\varkappa)^{1/\varkappa} - 1}{2r\varepsilon}\,,$$

which is exactly (6.3). Note that, by the Borell characterization, we do not lose generality by assuming that $\mu$ is full-dimensional (i.e., absolutely continuous).

From Lemma 6.1 we can now derive the following assertion.

**Lemma 6.2.** *Let $\mu$ be a $\varkappa$-concave probability measure on $\mathbf{R}^n$, $-\infty < \varkappa < 0$, and let $A$ be a Borel subset of $\mathbf{R}^n$ of measure $t = \mu(A) \leqslant \frac{1}{2}$. If $\rho > 0$ satisfies*

$$\mu\{|x| > \rho\} \leqslant \frac{t}{2}, \tag{6.6}$$

*then*

$$\mu^+(A) \geqslant \frac{c(\varkappa)}{\rho}\, t, \quad \text{where} \ \ c(\varkappa) = \frac{1 - (2/3)^{-\varkappa}}{-2\varkappa}. \tag{6.7}$$

*Proof.* By Lemma 6.1, since $\mu(B_\rho) \geqslant 1 - \frac{t}{2}$,

$$\begin{aligned} -2\rho\varkappa\, \mu^+(A) &\geqslant 1 - [\,t^{1-\varkappa} + (1-t)^{1-\varkappa}]\left(1 - \frac{t}{2}\right)^{\varkappa} \\ &= 1 - \left[t\left(\frac{t}{1-t/2}\right)^{-\varkappa} + (1-t)\left(\frac{1-t}{1-t/2}\right)^{-\varkappa}\right]. \end{aligned}$$

Clearly, on the interval $0 \leqslant t \leqslant 1/2$, the ratio $\frac{t}{1-t/2}$ is increasing and so bounded by $2/3$. Also $\frac{1-t}{1-t/2} \leqslant 1$, so

$$-2\rho\varkappa\, \mu^+(A) \geqslant 1 - \left[t\left(\frac{2}{3}\right)^{-\varkappa} + (1-t)\right] = t\left[1 - \left(\frac{2}{3}\right)^{-\varkappa}\right],$$

which is the claim (6.7). □

Note that $c(\varkappa)$ continuously depends on $\varkappa$ and $\lim_{\varkappa\to 0} c(\varkappa) = c(0) = \frac{1}{2}\log\frac{3}{2}$, while $c(\varkappa) \sim \frac{1}{-2\varkappa}$ as $\varkappa \to -\infty$. In particular, $c(\varkappa) \geqslant \frac{c}{1-\varkappa}$ for $\varkappa \leqslant 0$. As a result, we obtain the following assertion.

**Theorem 6.3.** *Let $\mu$ be a $\varkappa$-concave probability measure on $\mathbf{R}^n$, $-\infty < \varkappa < 0$, such that*

$$\int \Phi(|x|)\, d\mu(x) \leqslant D \tag{6.8}$$

*for some increasing continuous function $\Phi : [0, +\infty) \to [0, +\infty)$. For any Borel set $A$ in $\mathbf{R}^n$ of measure $t = \mu(A) \leqslant \frac{1}{2}$*

$$\mu^+(A) \geqslant \frac{c}{1-\varkappa}\, \frac{t}{\Phi^{-1}(\frac{2D}{t})}, \tag{6.9}$$

*where $c$ is a positive universal constant and $\Phi^{-1}$ is the inverse function.*

Indeed, by the Chebyshev inequality and the hypothesis (6.8),

$$\mu\{|x| > \rho\} \leqslant \frac{D}{\Phi(\rho)} \leqslant \frac{t}{2},$$

where the last bound is obviously fulfilled for $\rho \geqslant \Phi^{-1}(\frac{2D}{t})$. By Lemma 6.2, we get

$$\mu^+(A) \geqslant c(\varkappa)\,\frac{t}{\Phi^{-1}(\frac{2D}{t})},$$

and the theorem follows.

As a basic example, we consider the function $\Phi(x) = \exp\{(x/\lambda)^\alpha\}$ with parameters $\alpha, \lambda > 0$, which has the inverse $\Phi^{-1}(y) = \lambda \log^{1/\alpha} y$, $y \geqslant 1$. Then the hypothesis (6.8) with $D = 2$ is equivalent to saying that the Orlicz norm generated by the Young function $\psi_\alpha(x) = e^{|x|^\alpha} - 1$, $x \in \mathbf{R}$, is bounded by $\lambda$ for the Euclidean norm, i.e., $\|\,|x|\,\|_{\psi_\alpha} \leqslant \lambda$ in the Orlicz space $L^{\psi_\alpha}(\mathbf{R}^n, \mu)$.

**Corollary 6.4.** *Let $\mu$ be a $\varkappa$-concave probability measure on $\mathbf{R}^n$, $-\infty < \varkappa < 0$, such that, for some $\alpha > 0$ and $\lambda > 0$,*

$$\int \exp\Big\{\Big(\frac{|x|}{\lambda}\Big)^\alpha\Big\}\, d\mu(x) \;\leqslant\; 2. \tag{6.10}$$

*Then for any Borel set $A$ in $\mathbf{R}^n$ of measure $t = \mu(A) \leqslant \frac{1}{2}$ with some universal constant $c > 0$*

$$\mu^+(A) \;\geqslant\; \frac{c}{1-\varkappa}\,\frac{t}{\lambda \log^{1/\alpha}(4/t)}\,.$$

Now, we may recall Corollary 5.4.

**Corollary 6.5.** *Any $\varkappa$-concave probability measure $\mu$ on $\mathbf{R}^n$, $-\infty < \varkappa < 0$, such that*

$$\int \exp\Big\{\Big(\frac{|x|}{\lambda}\Big)^\alpha\Big\}\, d\mu(x) \leqslant 2, \quad \alpha, \lambda > 0,$$

*satisfies the weak Poincaré type inequality with rate function*

$$C(p) = C\lambda\,(1-\varkappa)\left(\frac{3}{2-p}\right)^{1/\alpha},$$

*where $C$ is a universal constant.*

## 7 Examples. Perturbation

Given a spherically invariant, absolutely continuous probability measure $\mu$ on $\mathbf{R}^n$, we write its density in the form

$$p(x) = \frac{1}{Z}\, e^{-V(|x|)}, \quad x \in \mathbf{R}^n,$$

where $V = V(t)$ is defined and finite for $t > 0$ and $Z$ is a normalizing factor. If $V$ is convex and nondecreasing, then $\mu$ is log-concave (and conversely). If not, one may hope that $\mu$ will be $\varkappa$-concave for some $\varkappa < 0$. Namely, by the Borell characterization (6.2) with $\varkappa < 0$, the $\varkappa$-concavity of $\mu$ is equivalent to the convexity of the function $p^{\varkappa_n}$, where $\varkappa_n = \frac{\varkappa}{1-n\varkappa}$. In other words, $\mu$ is $\varkappa$-concave if and only if

1) the function $V(t)$ is nondecreasing in $t > 0$;
2) the function $e^{-\varkappa_n V(t)}$ is convex on $(0, +\infty)$.

If $V$ is twice continuously differentiable, the second property is equivalent to

2') $V''(t) - \varkappa_n V'(t)^2 \geqslant 0$ for all $t > 0$

As a more specific example, we consider densities of the form

$$p(x) = \frac{1}{Z}\, e^{-(a+b|x|)^\alpha}, \quad x \in \mathbf{R}^n, \tag{7.1}$$

with parameters $a, b > 0$ and $\alpha > 0$, which corresponds to $V(t) = (a + bt)^\alpha$.

It is clear that property 1) is fulfilled. If $\alpha \geqslant 1$, $V$ is convex and the measure $\mu$ is log-concave. So, assume that $0 < \alpha < 1$, in which case $V$ is not convex. It is easy to verify, the inequality of property 2') holds for all $t > 0$ if and only it holds for $t = 0$, and then it reads as

$$(\alpha - 1) - \alpha\varkappa_n a^\alpha \geqslant 0.$$

Hence an optimal choice is $\varkappa_n = -\frac{1-\alpha}{\alpha a^\alpha}$ or, equivalently,

$$\varkappa = -\frac{1-\alpha}{\alpha a^\alpha - n(1-\alpha)} \quad \text{provided that} \quad \alpha a^\alpha - n(1-\alpha) > 0. \tag{7.2}$$

Conclusion 1. *The probability measure $\mu$ with density* (7.1) *is convex if and only if $\alpha a^\alpha - n(1-\alpha) \geqslant 0$, in which case it is $\varkappa$-concave with the convexity parameter $\varkappa$ given by* (7.2).

In other words, $\mu$ is convex only if the parameter $a$ is sufficiently large. By Corollary 6.5, if $\varkappa > -\infty$, i.e., if $\alpha a^\alpha - n(1-\alpha) > 0$, the measure $\mu$ satisfies the weak Poincaré type inequality

$$\|f - \mathbf{E}f\|_p \leqslant C(p)\, \|\nabla f\|_2, \quad 1 \leqslant p < 2, \tag{7.3}$$

with rate function

$$C(p) = C\left(\frac{3}{2-p}\right)^{1/\alpha}, \tag{7.4}$$

where $C$ depends on the parameters $a, b, \alpha$ and the dimension $n$.

However, it is unlikely that the requirement (7.2), $a > a_0 > 0$, is crucial for (7.3) to hold with some rate function. To see this, a perturbation argument may be used to prove the following elementary:

**Theorem 7.1.** *Assume that a metric probability space* $(M, d, \mu)$ *satisfies the weak Poincaré type inequality* (7.3)*. Let* $\nu$ *be a probability measure on* $M$*, which is absolutely continuous with respect to* $\mu$ *and has density* $w = \frac{d\nu}{d\mu}$ *such that*

$$c_1 \leqslant w(x) \leqslant c_2, \quad x \in M, \tag{7.5}$$

*for some* $c_1, c_2 > 0$*. Then* $(M, d, \nu)$ *also satisfies* (7.3) *with rate function* $C'(p) = \frac{2c_2}{\sqrt{c_1}}\, C(p)$*.*

*Proof.* Indeed, assume that $f$ is bounded and locally Lipschitz on $M$ with

$$\mathbf{E}f = \int f \, d\mu = 0.$$

Then, by (7.3) and (7.5), for any $p \in [1, 2)$

$$\begin{aligned} \|f\|_{L^p(\nu)}^p &= \int |f|^p \, d\nu \ \leqslant \ c_2 \int |f|^p \, d\mu \\ &\leqslant c_2 \, C(p)^p \Big( \int |\nabla f|^2 \, d\mu \Big)^{p/2} \ \leqslant \ \frac{c_2}{c_1^{p/2}} \, C(p)^p \Big( \int |\nabla f|^2 \, d\nu \Big)^{p/2}, \end{aligned}$$

so

$$\|f\|_{L^p(\nu)} \leqslant \frac{c_2^{1/p}}{c_1^{1/2}} \, C(p) \, \|\nabla f\|_{L^2(\nu)}.$$

Since $c_2 \geqslant 1$, we find

$$\inf_{c \in \mathbf{R}} \|f - c\|_{L^p(\nu)} \ \leqslant \ \frac{c_2}{\sqrt{c_1}} \, C(p) \, \|\nabla f\|_{L^2(\nu)}.$$

But, in general, $\|f - \mathbf{E}f\|_p \leqslant 2 \, \|f - c\|_p$ for any $c \in \mathbf{R}$. □

Let us return to the measure $\mu = \mu_a$ with density (7.2). Write $V_a(x) = (a + b|x|)^\alpha$ and write the normalizing constant as a function of $a$, $Z = Z(a)$, although it depends also on the remaining parameters $b > 0$ and $\alpha \in (0, 1)$. For all $a_1, a_2 \geqslant 0$ we have

$$|V_{a_1}(x) - V_{a_2}(x)| \ \leqslant \ |a_1 - a_2|^\alpha.$$

Therefore, the density $w(x) = \dfrac{d\mu_{a_1}(x)}{d\mu_{a_2}(x)}$ satisfies $c \leqslant w(x) \leqslant 1/c$ with

$$c = \frac{\min\{Z(a_1), Z(a_2)\}}{\max\{Z(a_1), Z(a_2)\}}\, e^{-|a_1-a_2|^\alpha},$$

so that the condition (7.4) is fulfilled. Hence, by Theorem 7.1, the weak Poincaré type inequality (7.3) holds for all measures $\mu_a$ simultaneously with rate function of the form (7.4), as long as it holds for at least one measure $\mu_a$. But, as we have already observed, the latter is true under (7.2) by the convexity property of such measures. Thus, Conclusion 1 may be complemented with the following one.

CONCLUSION 2. *Probability measures $\mu$ having densities* (7.1) *with arbitrary parameters $a, b \geqslant 0$, $\alpha \in (0,1)$ satisfy the weak Poincaré type inequality* (7.3) *with rate function $C(p) = C \cdot (\frac{3}{2-p})^{1/\alpha}$, where $C$ depends on $a$, $b$, $\alpha$, and $n$.*

## 8 Weak Poincaré with Oscillation Terms

Let us return to the setting of an abstract metric probability space $(M, d, \mu)$. It is now a good time to look at the relationship between the weak Poincaré type inequalities

$$\|f - \mathbf{E}f\|_p \leqslant C(p)\, \|\nabla f\|_2, \quad 1 \leqslant p < 2, \tag{8.1}$$

which is our main object of research, and Poincaré type inequalities

$$\mathrm{Var}_\mu(f) \leqslant \beta(s)\, \|\nabla f\|_2^2 + s\, \mathrm{Osc}\,(f)^2, \quad s > 0, \tag{8.2}$$

that involve an oscillation term $\mathrm{Osc}\,(f) = \operatorname{ess\,sup} f - \operatorname{ess\,inf} f$ and some nonnegative function $\beta(s)$. (Note that we always have $\mathrm{Var}_\mu(f) \leqslant \frac{1}{4}\, \mathrm{Osc}\,(f)^2$, so for $s \geqslant 1/4$ (8.2) is automatically fulfilled.)

In both cases, $f$ represents an arbitrary locally Lipschitz function with a possible reasonable constraint that the right-hand sides should be finite. Hence, from the point of view of direct applications, (8.2) makes sense only for bounded $f$, while (8.1) may also be used for many unbounded functions. Nevertheless, both forms are in a certain sense equivalent, i.e., there is some relationship between $C(p)$ and $\beta(s)$. To study this type of connections, we first note the following elementary inequality of Nash type.

**Theorem 8.1.** *Under the weak Poincaré type inequality* (8.1), *for all bounded locally Lipschitz $f$ on $M$ and any $p \in [1,2)$*

$$\mathrm{Var}_\mu(f) \leqslant C(p)^p\, \mathrm{Osc}\,(f)^{2-p}\, \|\nabla f\|_2^p. \tag{8.3}$$

Indeed, since (8.3) is translation invariant, we may assume $\mathbf{E}f = 0$. Then it is obvious that $\operatorname{ess\,inf} f \leqslant 0 \leqslant \operatorname{ess\,sup} f$, so $\mu$-almost everywhere $\mathrm{Osc}\,(f) \geqslant \|f\|_\infty \geqslant |f|$. By (8.1),

$$\mathbf{E}|f|^2 = \mathbf{E}|f|^p\ |f|^{2-p} \leqslant \mathbf{E}|f|^p\ \mathrm{Osc}\,(f)^{2-p} \leqslant C(p)^p (\mathbf{E}|\nabla f|^2)^{p/2}\,\mathrm{Osc}\,(f)^{2-p},$$

where all expectations are with respect to $\mu$.

From Theorem 8.1 we derive an additive form of (8.3).

**Theorem 8.2.** *Under the weak Poincaré type inequality* (8.1), (8.2) *holds with*

$$\beta(s) = \inf_{1 \leqslant p < 2} \left[ C(p)^2\, s^{1-\frac{2}{p}} \right]. \tag{8.4}$$

*Proof.* Using the Young inequality $xy \leqslant \frac{x^\alpha}{\alpha} + \frac{y^\beta}{\beta}$, where $x, y \geqslant 0$, $\alpha, \beta > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, for any $\varepsilon > 0$ we can estimate the right-hand side of (8.3) by

$$C(p)^p\ \left[ \frac{[\frac{1}{\varepsilon}\,(\mathbf{E}\,|\nabla f|^2)^{p/2}]^\alpha}{\alpha} + \frac{[\varepsilon\ \mathrm{Osc}\,(f)^{2-p}]^\beta}{\beta} \right].$$

Choose $\alpha = \frac{2}{p}$ and $\beta = \frac{2}{2-p}$, to get

$$\mathrm{Var}_\mu(f)\ \leqslant\ \frac{C(p)^p}{\alpha\varepsilon^\alpha}\ \mathbf{E}\,|\nabla f|^2 + \frac{C(p)^p\,\varepsilon^\beta}{\beta}\ \mathrm{Osc}\,(f)^2. \tag{8.5}$$

Put

$$s = \frac{C(p)^p\,\varepsilon^\beta}{\beta}, \quad \text{so that} \quad \varepsilon = \left[\frac{\beta s}{C(p)^p}\right]^{1/\beta}.$$

Then the coefficient in front of $\mathbf{E}\,|\nabla f|^2$ in (8.5) becomes

$$\frac{C(p)^p}{\alpha\varepsilon^\alpha} = \frac{C(p)^p}{\alpha}\ \left[\frac{\beta s}{C(p)^p}\right]^{-\alpha/\beta} = C(p)^{p(1+\frac{\alpha}{\beta})}\ \frac{1}{\alpha\beta^{\alpha/\beta}}\ s^{-\alpha/\beta}.$$

The first exponent on the right is

$$p(1 + \frac{\alpha}{\beta}) = p(1 + \frac{2}{p}\,\frac{2-p}{2}) = 2.$$

For the second term we have

$$\frac{1}{\alpha\beta^{\alpha/\beta}} = \frac{p}{2}\,\left(\frac{2-p}{2}\right)^{(2-p)/p}\ \leqslant 1.$$

Also $\frac{\alpha}{\beta} = \frac{2}{p} - 1$, and (8.5) yields $\mathrm{Var}_\mu(f) \leqslant C(p)^2\,s^{1-2/p}\,\mathbf{E}\,|\nabla f|^2 + s\ \mathrm{Osc}\,(f)^2$. $\square$

**Corollary 8.3.** *If for some $a, b \geqslant 0$ and $\alpha > 0$, the rate function in the weak Poincaré type inequality* (8.1) *admits the bound*

$$C(p) \leqslant a\left(\frac{b}{2-p}\right)^{1/\alpha}, \quad 1 \leqslant p < 2, \tag{8.6}$$

*then, with some numerical constants $\beta_0, \beta_1 > 0$,* (8.2) *holds with*

$$\beta(s) = \beta \log^{2/\alpha} \frac{1}{s}, \quad s > 0, \tag{8.7}$$

*where $\beta = \beta_0 a^2 \, (\beta_1 b)^{2/\alpha}$.*

*Proof.* We may and do assume that $s < \frac{1}{4}$. Write $p = 2-\varepsilon$, so that $0 < \varepsilon \leqslant 1$ and $\frac{2}{p} - 1 = \frac{\varepsilon}{2-\varepsilon}$. By Theorem 8.2, the hypothesis (8.6), and the inequality $\frac{\varepsilon}{2-\varepsilon} \leqslant \varepsilon$, for the optimal value of $\beta(s)$ we have

$$\beta(s) \leqslant C(p)^2 s^{1-\frac{2}{p}} \leqslant a^2 \, b^{2/\alpha} \, \frac{1}{\varepsilon^{2/\alpha}} \, \frac{1}{s^{\frac{\varepsilon}{2-\varepsilon}}} \leqslant a^2 \, b^{2/\alpha} \, \frac{1}{\varepsilon^{2/\alpha} s^{\varepsilon}}$$

for all $\varepsilon \in (0,1]$. To optimize over all such $\varepsilon$, we consider the function $\varphi(\varepsilon) = \varepsilon^{2/\alpha} s^{\varepsilon}$. Then $\varphi(0) = 0$, $\varphi(1) = s$, and $\varphi'(\varepsilon) = \varepsilon^{2/\alpha} s^{\varepsilon} \, (\frac{2}{\alpha\varepsilon} - \log \frac{1}{s})$. Hence the (unique) point of maximum of $\varphi$ on $[0,+\infty)$ is $\varepsilon_0 = \frac{2}{\alpha \log \frac{1}{s}}$ and, at this point,

$$\varphi(\varepsilon_0) = \left(\frac{2}{\alpha \log \frac{1}{s}}\right)^{2/\alpha} e^{-2/\alpha} = \left(\frac{2}{\alpha e}\right)^{2/\alpha} \frac{1}{\log^{2/\alpha} \frac{1}{s}}.$$

Hence, if $\varepsilon_0 \leqslant 1$, i.e., $s \leqslant e^{-2/\alpha}$, then

$$\beta(s) \leqslant a^2 \, b^{2/\alpha} \frac{1}{\varphi(\varepsilon_0)} = a^2 \, b^{2/\alpha} \left(\frac{\alpha e}{2}\right)^{2/\alpha} \log^{2/\alpha} \frac{1}{s} \leqslant e^{1/e} a^2 (be)^{2/\alpha} \log^{2/\alpha} \frac{1}{s} \, .$$

Note that, since $s < 1/4$, the requirement $s \leqslant e^{-2/\alpha}$ is automatically fulfilled, as long as $\alpha \geqslant 1/\log 2$. In that case, (8.7) is thus proved with constants $\beta_0 = e^{1/e}$ and $\beta_1 = e$.

Now, let $\alpha < 1/\log 2$ and $s \geqslant e^{-2/\alpha}$. Then $\varphi$ is increasing and is maximized on [0,1] at $\varepsilon = 1$, which gives $\beta(s) \leqslant a^2 \, b^{2/\alpha} \, \frac{1}{s}$. So, we need the bound

$$\frac{1}{s} \leqslant A \log^{2/\alpha} \frac{1}{s}$$

in the interval $e^{-2/\alpha} \leqslant s \leqslant 1/4$. Since the function $t \log \frac{1}{t}$ is decreasing in $t \geqslant 1/e$, the optimal value of $A$ is attained at $s = 1/4$, so $A = 4/\log^{2/\alpha} 4$. Therefore, (8.7) is valid with $\beta_0 = 4\, e^{1/e}$ and $\beta_1 = e/\log 4$. Corollary 8.3 is proved. □

In particular, we have the following assertion.

**Corollary 8.4.** *Any $\varkappa$-concave probability measure $\mu$ on $\mathbf{R}^n$, $-\infty < \varkappa < 0$, such that $\int \exp\{(\frac{|x|}{\lambda})^\alpha\}\, d\mu(x) \leqslant 2$ $(\alpha, \lambda > 0)$, satisfies* (8.2) *with*

$$\beta(s) = \beta \log^{2/\alpha} \frac{1}{s}, \quad s > 0,$$

*where $\beta = \beta_0 \lambda^2 (1-\varkappa)^2\, \beta_1^{2/\alpha}$, $\beta_0, \beta_1 > 0$ are numerical constants.*

On the basis of (8.1) one may also consider a more general type of "oscillations," for example, Poincaré type inequalities of the form

$$\operatorname{Var}_\mu(f) \;\leqslant\; \beta_q(s)\, \|\nabla f\|_2^2 + s\, \|f - \mathbf{E}f\|_q^2, \quad s > 0, \tag{8.8}$$

with a fixed finite parameter $q > 2$. As we will see, this form is natural in the study of the slow rates of convergence of the associated semigroups $P_t f$, when $f$ is unbounded, but is still in $L^q(\mu)$. Note that (8.8) is automatically fulfilled for $s \geqslant 1$ (since $\beta_q$ is nonnegative), so one may restrict oneself to the values $s < 1$. We prove the following assertion.

**Theorem 8.5.** *Under the weak Poincaré type inequality* (8.1) *with rate function $C(p)$,* (8.8) *holds with*

$$\beta_q(s) = \inf_{1 \leqslant p < 2} \left[ C(p)^2\; s^{-\frac{q}{q-2}\frac{2-p}{p}} \right]. \tag{8.9}$$

*Proof.* The argument is very similar to the one used in the proof of Theorem 8.2. Given $p \in [1,2)$ and $q > 2$, by the Hölder inequality, we have

$$\mathbf{E}\, |f|^2 \leqslant \|f\|_p^r\, \|f\|_q^{2-r},$$

where $r = \frac{p(q-2)}{q-p}$. Therefore, if $\mathbf{E}f = 0$ (which we assume), by the hypothesis (8.1),

$$\mathbf{E}\, |f|^2 \;\leqslant\; C(p)\, \|f\|_q^{2-r}\, \|\nabla f\|_2^r. \tag{8.10}$$

Using the Young inequality with exponents $\alpha, \beta > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, for any $\varepsilon > 0$ we can estimate the right-hand side of (8.10) by

$$C(p)^r \;\left[ \frac{[\frac{1}{\varepsilon}\, \|\nabla f\|_2^r]^\alpha}{\alpha} + \frac{[\varepsilon\, \|f\|_q^{2-r}]^\beta}{\beta} \right].$$

Choose $\alpha = \frac{2}{r}$ and $\beta = \frac{2}{2-r}$ to get

$$\mathbf{E}\,|f|^2 \leqslant \frac{C(p)^r}{\alpha\varepsilon^\alpha}\,\mathbf{E}\,|\nabla f|^2 + \frac{C(p)^r\,\varepsilon^\beta}{\beta}\,\|f\|_q^2. \tag{8.11}$$

Put

$$s = \frac{C(p)^r\,\varepsilon^\beta}{\beta}, \quad \text{so that} \quad \varepsilon = \left[\frac{\beta s}{C(p)^r}\right]^{1/\beta}.$$

Then the coefficient in front of $\mathbf{E}\,|\nabla f|^2$ in (8.11) becomes

$$\frac{C(p)^r}{\alpha\varepsilon^\alpha} = \frac{C(p)^r}{\alpha}\left[\frac{\beta s}{C(p)^r}\right]^{-\alpha/\beta} = C(p)^{r(1+\frac{\alpha}{\beta})}\,\frac{1}{\alpha\beta^{\alpha/\beta}}\,s^{-\alpha/\beta}.$$

The first exponent on the right is $r(1+\frac{\alpha}{\beta}) = r(1+\frac{2}{r}\,\frac{2-r}{2}) = 2$. For the second term we have

$$\frac{1}{\alpha\beta^{\alpha/\beta}} = \frac{r}{2}\left(\frac{2-r}{2}\right)^{(2-r)/r} \leqslant 1.$$

Also $\frac{\alpha}{\beta} = \frac{2}{r} - 1 = \frac{q}{q-2}\,\frac{2-p}{p}$, and we arrive at

$$\mathbf{E}\,|f|^2 \leqslant C(p)^2\,s^{-\frac{q}{q-2}\,\frac{2-p}{p}}\,\mathbf{E}\,|\nabla f|^2 + s\,\|f\|_q^2,$$

which is the claim. □

Now, we can strengthen Corollaries 8.3 and 8.4.

**Corollary 8.6.** *If the rate function in the weak Poincaré type inequality* (8.1) *admits the bound* (8.6)*, then* (8.8) *holds with*

$$\beta_q(s) = \beta\log^{\frac{2}{\alpha}}\frac{2}{s}, \quad s > 0, \tag{8.12}$$

*where* $\beta = 2a^2\,(4b\,\frac{q}{q-2})^{2/\alpha}$.

*Proof.* As in the proof of Corollary 8.3, we assume that $s < 1$ and write $p = 2-\varepsilon$, so that $0 < \varepsilon \leqslant 1$ and $\frac{2}{p} - 1 = \frac{\varepsilon}{2-\varepsilon}$. Put $Q = \frac{q}{q-2}$. By Theorem 8.5 and the inequality $\frac{\varepsilon}{2-\varepsilon} \leqslant \varepsilon$, for the optimal value of $\beta_q(s)$ we have

$$\beta_q(s) \leqslant C(p)^2\,s^{Q(1-\frac{2}{p})} \leqslant a^2\,b^{2/\alpha}\,\frac{1}{\varepsilon^{2/\alpha}}\,\frac{1}{s^{Q\frac{\varepsilon}{2-\varepsilon}}} \leqslant a^2\,b^{2/\alpha}\,\frac{1}{\varepsilon^{2/\alpha}s^{Q\varepsilon}}$$

for all $\varepsilon \in (0,1]$. To optimize over all such $\varepsilon$, we consider the function $\varphi(\varepsilon) = \varepsilon^{2/\alpha}s^{Q\varepsilon}$. We have $\varphi(0) = 0$ and $\varphi(1) = s^Q$. As we know, the (unique) point of maximum of $\varphi$ on $[0,+\infty)$ is $\varepsilon_0 = \frac{2}{Q\alpha\log\frac{1}{s}}$ and, at this point,

$$\varphi(\varepsilon_0) = \left(\frac{2}{Q\alpha\log\frac{1}{s}}\right)^{2/\alpha}e^{-2/\alpha} = \left(\frac{2}{Q\alpha e}\right)^{2/\alpha}\frac{1}{\log^{2/\alpha}\frac{1}{s}}.$$

Hence, if $\varepsilon_0 \leqslant 1$, i.e., $s \leqslant e^{-2/Q\alpha}$, then

$$\beta_q(s) \leqslant \frac{a^2\, b^{2/\alpha}}{\varphi(\varepsilon_0)} = a^2\, b^{2/\alpha} \left(\frac{Q\alpha e}{2}\right)^{2/\alpha} \log^{2/\alpha} \frac{1}{s} \leqslant 2a^2\, (Qbe)^{2/\alpha} \log^{2/\alpha} \frac{1}{s}\,,$$

where we used $(\frac{\alpha}{2})^{2/\alpha} \leqslant e^{1/e} < 2$. Thus, for this range of $s$, (8.12) is proved.

Now, we assume that $s \geqslant e^{-2/Q\alpha}$. Then $\varphi$ is increasing and is maximized on [0,1] at $\varepsilon = 1$, which gives $\beta_q(s) \leqslant a^2\, b^{2/\alpha} s^{-Q}$. So, we need a bound of the form

$$s^{-Q} \leqslant A \log^{2/\alpha}(2/s)$$

or, equivalently,

$$A^{-1/Q} \leqslant s \log^{2/Q\alpha}(2/s)$$

in the interval $e^{-2/Q\alpha} \leqslant s \leqslant 1$. The function $s \log^c(2/s)$ with parameter $c > 0$ is increasing in $0 < s \leqslant 2e^{-c}$ and decreasing in $s \geqslant 2e^{-c}$, so we only need to consider the endpoints of that interval. For the point $s = 1$ we get

$$A = 1/\log^{2/\alpha} 2,$$

while for $s = e^{-2/Q\alpha}$ we get

$$A = \frac{e^{2/\alpha}}{\log^{2/\alpha}(2\, e^{2/Q\alpha})} \leqslant \left(\frac{e}{\log 2}\right)^{2/\alpha} < 4^{2/\alpha}.$$

The corollary is proved. □

Remark 8.1. As a result, one may also generalize Corollary 8.4. Namely, any $\varkappa$-concave probability measure $\mu$ on $\mathbf{R}^n$ with $\varkappa < 0$ and

$$\int \exp\Big\{\Big(\frac{|x|}{\lambda}\Big)^{\alpha}\Big\}\, d\mu(x) \leqslant 2, \quad \alpha, \lambda > 0,$$

satisfies the weak Poincaré type inequality (8.8) with

$$\beta_q(s) = \beta \log^{2/\alpha} \frac{2}{s}, \quad q > 2,$$

where $\beta$ depends on $\lambda$, $\alpha$, $\varkappa$, and $q$.

Remark 8.2. It is also possible to derive a weak Poincaré type inequality (8.1) from (8.2) or (8.8) with some rate functions $C(p)$ explicitly in terms of $\beta(s)$ or $\beta_q(s)$. This may be done by virtue of the measure capacity inequalities

$$\mathrm{cap}_\mu(A) \geqslant J(\mu(A)),$$

which we discussed in Section 5. As was shown in [4], the latter is fulfilled with $J(t) = t/(4\beta(t/4))$ in the presence of (8.2). Hence, applying Theorem

5.7 in a somewhat weaker form (5.15), we conclude that (8.1) holds with

$$C(p)^2 = 81 \sup_{0<t\leqslant 1/2} \left[ \frac{t^{\frac{4}{2+p}}}{J(t)} \right] = 81 \sup_{0<t\leqslant 1/2} \left[ 4^{\frac{4}{2+p}}\, t^{\frac{2-p}{2+p}}\, \beta(t/4) \right].$$

Hence we arrive at the following assertion.

**Theorem 8.7.** *In the presence of* (8.2)*, the weak Poincaré type inequality* (8.1) *holds with rate function given by*

$$C(p)^2 = C^2 \sup_{0<s\leqslant 1/8} \left[ s^{\frac{2-p}{2+p}}\, \beta(s) \right],$$

*where $C$ is a universal constant.*

## 9 Convergence of Markov Semigroups

Let $\mu$ be an absolutely continuous Borel probability measure on $\mathbf{R}^n$. We assume that the measure is regular enough in the following sense: There exists a family of operators $(P_t)_{t\geqslant 0}$, acting on some space $\mathcal{D}$ of bounded smooth functions $f$ on $\mathbf{R}^n$ with bounded partial derivatives, dense in all $L^p(\mu)$, $p \geqslant 1$, such that

1) $P_t f \in \mathcal{D}$ for all $f \in \mathcal{D}$,
2) $P_0$ is the identity operator, i.e., $P_0 f = f$ for all $f \in \mathcal{D}$,
3) $P_t$ forms a semigroup, i.e., $P_t(P_s f) = P_{t+s} f$ for all $t, s \geqslant 0$,
4) for any $f \in \mathcal{D}$, in the space $L^\infty(\mu)$, we have $\|P_t f - f\|_\infty \to 0$ as $t \to 0^+$,
5) for any $f \in \mathcal{D}$, in the space $L^1(\mu)$, the limit $Lf = \lim_{t\to 0^+} \frac{P_t f - f}{t}$ exists,
6) for all $f, g \in \mathcal{D}$

$$\int \langle \nabla f, \nabla g \rangle \, d\mu = -\int f\, Lg \, d\mu. \tag{9.1}$$

Equality in 5) expresses the property that $L$ represents the generator of the semigroup $P_t$. This is usually denoted by $P_t = e^{tL}$, where the exponential function is understood in the operator sense. Owing to 1) and 3), it may be generalized as the property that for any $f \in \mathcal{D}$ and $t \geqslant 0$, in the space $L^1(\mu)$,

$$L(P_t f) = \lim_{\varepsilon\to 0^+} \frac{P_{t+\varepsilon} f - P_t f}{\varepsilon}. \tag{9.2}$$

In other words, the $L^1$-valued map $t \to P_t f$ is differentiable from the right and has the right derivative $L(P_t f)$. The equalities (9.1) and (9.2) may be used to prove, in particular, the following assertion.

**Lemma 9.1.** *Given a twice continuously differentiable function $u$ on the real line, for any $f \in \mathcal{D}$ the function $t \to \int u(P_t f)\, d\mu$ is differentiable from the right and has the right derivative*

$$\frac{d}{dt} \int u(P_t f)\, d\mu = - \int u''(P_t f)\, |\nabla P_t f|^2\, d\mu. \tag{9.3}$$

To illustrate classical applications, we assume that a measure $\mu$ satisfies a Poincaré type inequality

$$\lambda_1 \operatorname{Var}_\mu(f) \leqslant \int |\nabla f|^2\, d\mu \tag{9.4}$$

for some $\lambda_1 > 0$ in the class of all smooth $f$ on $\mathbf{R}^n$.

For $u(x) = x$ the equality (9.3) implies that the function $\varphi(t) = \int P_t f\, d\mu$, where $f \in \mathcal{D}$, has the right derivative zero at every point $t \geqslant 0$. Since this function is also continuous, it must be equal to a constant, i.e., $\int f\, d\mu$. Taking $u(x) = x^2$ and assuming that $\int f\, d\mu = 0$, from (9.3) and (9.4) we have

$$\frac{d}{dt} \int |P_t f|^2\, d\mu = -2 \int |\nabla P_t f|^2\, d\mu \leqslant -2\lambda_1 \int |P_t f|^2\, d\mu.$$

Thus, the function $\varphi(t) = \int |P_t f|^2\, d\mu$ is continuous and has the right derivative satisfying $\varphi'(t) \leqslant -2\lambda_1 \varphi(t)$. It is a simple calculus exercise to derive from this differential inequality the bound on the rate of convergence, $\varphi(t) \leqslant \varphi(0) e^{-2\lambda_1 t}$. Therefore,

$$\int |P_t f|^2\, d\mu \leqslant e^{-2\lambda_1 t} \int |f|^2\, d\mu, \quad t \geqslant 0. \tag{9.5}$$

In particular, we obtain a contraction property $\|P_t f\|_2 \leqslant \|f\|_2$ for all $f \in \mathcal{D}$, which allows us to extend $P_t$ to all $L^2(\mu)$ as a linear contraction. Moreover, by continuity, (9.5) extends to all $f \in L^2(\mu)$ with $\mu$-mean zero, and we also have

$$\int P_t f\, d\mu = \int f\, d\mu.$$

Our next natural step is to generalize (9.5) to $L^p$-spaces.

**Theorem 9.2.** *For all $f \in L^p(\mu)$, $p > 1$, and $t \geqslant 0$*

$$\iint |P_t f(x) - P_t f(y)|^p\, d\mu(x) d\mu(y) \leqslant e^{-\frac{4(p-1)}{p} \lambda_1 t} \iint |f(x) - f(y)|^p\, d\mu(x) d\mu(y). \tag{9.6}$$

*Proof.* As in the previous example of the quadratic function, for any twice continuously differentiable, convex function $u$ on the real line and any $f \in \mathcal{D}$, by (9.3) and (9.4), we have

$$\frac{d}{dt}\int u(P_t f)\,d\mu = -\int u''(P_t f)\,|\nabla P_t f|^2\,d\mu$$
$$= -\int |\nabla v(P_t f)|^2\,d\mu \ \leqslant\ -\lambda_1 \operatorname{Var}_\mu[v(P_t f)], \qquad (9.7)$$

where the derivative is understood as the derivative from the right and $v$ is a differentiable function satisfying $v'^2 = u''$. In particular, we may take $u(z) = |z|^p$ with $p \geqslant 2$, so that $u''(z) = p(p-1)|z|^{p-2}$ and

$$v(z) = 2\sqrt{\frac{p-1}{p}}\ \operatorname{sign}(z)\,|z|^{p/2},$$

to get

$$\frac{d}{dt}\int |P_t f|^p\,d\mu \ \leqslant\ -4\lambda_1\,\frac{p-1}{p}\int |P_t f|^p\,d\mu \qquad (9.8)$$

provided that

$$\int \operatorname{sign}(P_t f)\,|P_t f|^{p/2}\,d\mu = 0.$$

The last equality holds, for example, when $P_t f$ has a distribution under $\mu$, symmetric about zero. Moreover, a slight modification of $u(z) = |z|^p$ near zero allows us to replace the constraint $p \geqslant 2$ in (9.7) and (9.8) by the weaker condition $p > 1$ (cf. details at the end of the proof).

Now, on $M = \mathbf{R}^n \times \mathbf{R}^n$, we consider the product measure $\mu \otimes \mu$. By the subadditivity property of the variance functional, it also satisfies the Poincaré type inequality (9.4) with the same constant $\lambda_1$. In addition, with this measure one may associate the semigroup $\overline{P}_t$, $t \geqslant 0$, acting on a certain space $\overline{\mathcal{D}}$ of bounded smooth functions on $\mathbf{R}^n \times \mathbf{R}^n$ with bounded partial derivatives containing functions of the form

$$\overline{f}(x,y) = f(x) - f(y), \quad x,y \in \mathbf{R}^n, \quad f \in \mathcal{D}.$$

It easy to see that for such functions

$$(\overline{P}_t\overline{f})(x,y) = P_t f(x) - P_t f(y), \quad (\overline{Lf})(x,y) = Lf(x) - Lf(y),$$

where $\overline{L}$ is the generator of $\overline{P}_t$. Apply (9.8) to the product space $(\mathbf{R}^n \times \mathbf{R}^n, \mu \otimes \mu)$. Since $\overline{P}_t\overline{f}$ has a symmetric distribution under $\mu \otimes \mu$ about zero, the function

$$\varphi(t) = \iint |P_t f(x) - P_t f(y)|^p\,d\mu(x)d\mu(y)$$

is continuous and has the right derivative at every point $t \geqslant 0$ satisfying the differential inequality

$$\varphi'(t) \leqslant -C\,\varphi(t) \tag{9.9}$$

with $C = 4\lambda_1 \frac{p-1}{p}$. Then $\varphi(t) \leqslant \varphi(0)e^{-Ct}$, which is the claim.

Thus, when $p \geqslant 2$, every $P_t$ represents a continuous linear operator on $\mathcal{D}$ with respect to the $L^p$-norm, so it may be extended to the whole $L^p(\mu)$; moreover, the inequality (9.6) remains valid for all functions $f$ in $L^p(\mu)$.

Now, let us see what modifications may be made in the case $1 < p < 2$. Given a fixed natural number $N$, define a convex, twice continuously differentiable, even function $u_N$ through its second derivative

$$u_N''(z) = p(p-1)\,\min\{|z|^{p-2}, N\}, \quad z \in \mathbf{R},$$

and by requiring that $u_N(0) = u_N'(0) = 0$. Also, define an odd function $v_N$ through its first derivative

$$v_N'(z) = \mathrm{sign}(z)\,\sqrt{u_N''(z)} = \mathrm{sign}(z)\,\sqrt{p(p-1)}\,\min\left\{|z|^{\frac{p}{2}-1}, \sqrt{N}\right\}, \quad z \neq 0,$$

or, equivalently,

$$v_N(z) = \sqrt{p(p-1)}\int_0^z \min\{|y|^{\frac{p}{2}-1}, \sqrt{N}\}\,dy.$$

We note that $v_N$ is differentiable everywhere, except for $z = 0$, at which point the left and right derivatives exist, but do not coincide. On the other hand, $|v_N'(z)|$ is continuous everywhere, including the origin point $z = 0$, so that, in the class of all smooth $g$ on $\mathbf{R}^n$, we always have a chain rule

$$u_N''(g(x))|\nabla g(x)|^2 = |\nabla v_N(g(x))|^2, \quad x \in \mathbf{R}^n,$$

even if $g(x) = 0$. Thus, the first part of (9.7) remains valid for $u_N$, i.e.,

$$\frac{d}{dt}\int u_N(P_t f)\,d\mu = -\int |\nabla v_N(P_t f)|^2\,d\mu.$$

We also recall that the Poincaré type inequality (9.4) extends to all locally Lipschitz functions $f$ on $\mathbf{R}^n$. In particular, by the chain rule,

$$\lambda_1\,\mathrm{Var}_\mu(T(g)) \leqslant \int |T'(g)|^2\,|\nabla g|^2\,d\mu$$

if $g$ is smooth on $\mathbf{R}^n$ and $T$ on $\mathbf{R}$. For a fixed $g$ this inequality may be written in dimension one as

$$\lambda_1\,\mathrm{Var}_\nu(T) \leqslant \int |T'|^2\,d\pi$$

with respect to the distribution $\nu$ of $g$ under $\mu$ and the distribution $\pi$ of $g$ under the finite measure $|\nabla g|^2\,d\mu$. At this step, it is only required that $T$ be locally Lipschitz on the line, and this is indeed true for $T = v_N$. Therefore, the second part of (9.7) also holds for $v_N$, and we get

$$\frac{d}{dt}\int u_N(P_t f)\,d\mu \leqslant -\lambda_1 \int v_N(P_t f)^2\,d\mu \tag{9.10}$$

provided that

$$\int v_N(P_t f)\,d\mu = 0.$$

Now, to estimate further the right-hand side of (9.10), we use the integral description of $v_N$ to see that for $z > 0$

$$\begin{aligned} 0 \leqslant\ v(z) - v_N(z) &= \sqrt{p(p-1)}\int_0^z \Big[y^{\frac{p}{2}-1} - \min\{y^{\frac{p}{2}-1}, \sqrt{N}\,\}\Big]\,dy \\ &\leqslant \sqrt{p(p-1)}\int_0^{+\infty} y^{\frac{p}{2}-1}\,1_{\{y^{\frac{p}{2}-1} > \sqrt{N}\,\}}\,dy \\ &= 2\sqrt{\frac{p-1}{p}}\,N^{-\frac{p}{2(2-p)}}. \end{aligned}$$

Hence

$$v(z)^2 - v_N(z)^2 \leqslant 2\,v(z)(v(z) - v_N(z)) \leqslant \frac{8(p-1)}{p}\,z^{\frac{p}{2}}\,N^{-\frac{p}{2(2-p)}} \leqslant \frac{4}{\sqrt{N}}\,z^{\frac{p}{2}},$$

so that

$$v_N(z)^2 \geqslant v(z)^2 - \frac{4}{\sqrt{N}}\,z^{\frac{p}{2}} = \frac{4(p-1)}{p}\,u(z) - \frac{4}{\sqrt{N}}\,z^{\frac{p}{2}},$$

and thus, for all $z \in \mathbf{R}$,

$$v_N(z)^2 \geqslant \frac{4(p-1)}{p}\,u_N(z) - \frac{4}{\sqrt{N}}\,|z|^{\frac{p}{2}}.$$

Therefore, (9.10) may be continued as

$$\frac{d}{dt}\int u_N(P_t f)\,d\mu \leqslant -4\lambda_1\,\frac{p-1}{p}\int u_N(P_t f)\,d\mu + \frac{4}{\sqrt{N}}\int |P_t f|^{\frac{p}{2}}\,d\mu,$$

where we assumed that $\int v_N(P_t f)\,d\mu = 0$. Since $f$ is bounded, all $P_t f$ are uniformly bounded (cf. Corollary 9.3 concerning large values of $p$), so the above estimate yields

$$\frac{d}{dt}\int u_N(P_t f)\,d\mu \;\leqslant\; -4\lambda_1\,\frac{p-1}{p}\int u_N(P_t f)\,d\mu + \frac{A}{\sqrt{N}} \tag{9.11}$$

with some constant $A$ independent of $t$. Applying (9.11) in the product space to functions of the form $f(x) - f(y)$, as in the case $p \geqslant 2$, we find that the function

$$\varphi_N(t) = \iint u_N(P_t f(x) - P_t f(y))\,d\mu(x)d\mu(y)$$

is continuous and has the right derivative satisfying at every point $t \geqslant 0$ the following modified form of (9.9):

$$\varphi_N'(t) \;\leqslant\; -C\varphi_N(t) + \varepsilon_N,$$

where $\varepsilon_N = \frac{A}{\sqrt{N}}$ and $C = 4\lambda_1\,\frac{p-1}{p}$, as above. In terms of $\psi_N(t) = \varphi_N(t)\,e^{Ct}$ this differential inequality takes a simpler form $\psi_N'(t) \leqslant \varepsilon_N\,e^{Ct}$, which is easily solved as

$$\psi_N(t) \leqslant \psi_N(0) + \frac{\varepsilon_N}{C}\,(e^{Ct} - 1).$$

Equivalently,

$$\varphi_N(t) \leqslant \varphi_N(0)\,e^{-Ct} + \frac{\varepsilon_N}{C}\,(1 - e^{-Ct}),$$

so

$$\iint u_N(P_t f(x) - P_t f(y))\,d\mu(x)d\mu(y) \leqslant e^{-\frac{4(p-1)}{p}\,\lambda_1 t}\iint u_N(f(x) - f(y))\,d\mu(x)d\mu(y) + \frac{\varepsilon_N}{C}.$$

It remains to let $N \to \infty$ and use the property that $u_N \to u$ uniformly on bounded intervals of the line. Thus, (9.6) holds for all functions $f$ in $\mathcal{D}$ and therefore for all $f$ from the whole space $L^p(\mu)$. Theorem 9.2 is proved. □

REMARK 9.1. Let us describe several immediate applications of Theorem 9.2.

1. Thus, every $P_t$ represents a linear contraction in $L^p(\mu)$. Note that if $\int f\,d\mu = 0$, by the Jensen inequality, the left-hand side of (9.6) majorizes $\|P_t f\|_p^p$ and the integral on the right-hand side is majorized by $2^p\|f\|_p^p$. Hence we get a hypercontractive inequality

$$\|P_t f\|_p \;\leqslant\; 2e^{-\frac{4(p-1)}{p^2}\,\lambda_1 t}\,\|f\|_p.$$

2. Similarly, one may consider Orlicz norms different from $L^p$-norms. For example, using the Taylor expansion for $\psi_2(z) \equiv e^{z^2} - 1$, from (9.6) we get that for any $\alpha > 0$

$$\iint \psi_2(\alpha|P_t f(x) - P_t f(y)|)\, d\mu(x) d\mu(y) \leqslant e^{-2\lambda_1 t} \iint \psi_2(\alpha|f(x) - f(y)|)\, d\mu(x) d\mu(y).$$

Hence the operator $P_t$ continuously acts on $L^{\psi_2}(\mu)$.

3. Letting $p \to +\infty$ in (9.6), we conclude that for any bounded measurable function $f$ on $\mathbf{R}^n$ and for any $t \geqslant 0$

$$\mathrm{Osc}\,(P_t f) \leqslant \mathrm{Osc}\,(f). \tag{9.12}$$

In particular, $P_t$ represents a contraction in $L^\infty(\mu)$, while for finite $p > 1$ these operators are hypercontractive.

4) Since the inequality (9.12) does not involve $\lambda_1$, it remains valid in the case $\lambda_1 = 0$. Such properties may be seen with the help of Lemma 9.1. Namely, from (9.3) it follows that, if $u$ is additionally convex, then the function $t \to \int u(P_t f)\, d\mu$ is nonincreasing, so that

$$\int u(P_t f)\, d\mu \leqslant \int u(f)\, d\mu. \tag{9.13}$$

For example, the case $u(z) = |z|^p$, $p > 1$, yields

$$\|P_t f\|_p \leqslant \|f\|_p. \tag{9.14}$$

By the continuity of $P_t$ on $L^p$, this inequality extends from $\mathcal{D}$ to the whole space $L^p(\mu)$. Note that, in Lemma 9.1, it is assumed that $u$ is twice continuously differentiable and this is fulfilled as long as $p \geqslant 2$. However, the range $1 < p \leqslant 2$ may be treated with the help of a smooth approximation, such as in the proof of Theorem 9.2. Moreover, (9.14) remains valid for $p = 1$. We also note that, applying (9.14) in product spaces with $p = +\infty$, we arrive at (9.12).

## 10 Markov Semigroups and Weak Poincaré

As the next natural step, one may wonder what a weak Poincaré type inequality

$$\|f - \mathbf{E}f\|_p \leqslant C(p)\, \|\nabla f\|_2, \quad 1 \leqslant p < 2, \tag{10.1}$$

is telling us about possible contractivity property of the semigroup $(P_t)_{t \geqslant 0}$ associated to the Borel probability measure $\mu$ on $\mathbf{R}^n$. As in the previous section, we assume that properties 1)–6) are fulfilled, so that one may develop analysis, such as the basic identity (9.3) of Lemma 9.1.

Since (10.1) is weaker than the usual Poincaré type inequality (9.4), it is natural to expect to get a weak version of Theorem 9.2 on the rate of convergence of $P_t f$ to the constant function. In the classical case $p = 2$, lower rate of convergence have been studied by many authors. In particular, for this aim, developing the ideas of Ligget [27], Röckner and Wang [35] proposed to use a weak Poincaré type inequality with the generalized “oscillation term”

$$\mathrm{Var}_\mu(f) \ \leqslant\ \beta(s)\,\|\nabla f\|_2^2 + s\,\Phi(f)^2, \quad s > 0, \tag{10.2}$$

where $\Phi$ is a nonnegative functional on $\mathcal{D}$ satisfying

$$\Phi(P_t f) \leqslant \Phi(f) \quad \text{for all} \ \ t \geqslant 0. \tag{10.3}$$

Indeed, by Lemma 9.1, applied to $u(z) = z^2$, we have

$$\frac{d}{dt}\int |P_t f|^2\,d\mu = -2\int |\nabla P_t f|^2\,d\mu.$$

Hence, by (10.2) and (10.3), if $\int f\,d\mu = 0$, the function $\varphi(t) = \int |P_t f|^2\,d\mu$ has the right derivative satisfying

$$\varphi'(t) \ \leqslant\ -\frac{2}{\beta(s)}\,\varphi(t) + \frac{2s}{\beta(s)}\,\Phi(f)^2.$$

This differential inequality is solved as

$$\varphi(t) \leqslant \varphi(0)\,e^{-2t/\beta(s)} + s\,(1 - e^{-2t/\beta(s)})\,\Phi(f)^2,$$

so

$$\int |P_t f|^2\,d\mu \ \leqslant\ \inf_{s>0}\left[e^{-2t/\beta(s)}\int |f|^2\,d\mu + s\,\Phi(f)^2\right]. \tag{10.4}$$

Thus, we get a more general statement on the rate of convergence than the classical inequality (9.5), when $\beta(s) = 1/\lambda_1$, which is obtained from (10.4) by letting $s \to 0$. In applications, the right-hand side of (10.4) can be simplified as

$$\int |P_t f|^2\,d\mu \ \leqslant\ \xi(t)\left[\frac{1}{2}\int |f|^2\,d\mu + \Phi(f)^2\right], \tag{10.5}$$

where $\xi(t) = \inf\{s > 0 : \beta(s)\log\frac{2}{s} \leqslant 2t\}$.

As the most interesting examples, one may apply this scheme to the functionals $\Phi(f) = \mathrm{Osc}\,(f)$, or more generally $\Phi(f) = \|f - \mathbf{E}f\|_q$ or just $\Phi(f) = \|f\|_q$. Then, by the continuity of $P_t$, the resulting inequalities (10.4) and (10.5) extend from $\mathcal{D}$ to $L^q$-spaces.

In the presence of (10.1), we look for a corresponding expression for the bound on the rate of convergence explicitly in terms of the function $C(p)$. For this aim, we may appeal to Theorem 8.5, which relates (10.1) to (10.2) in the case $\Phi(f) = \|f - \mathbf{E}f\|_q$, $q > 2$. Indeed, by (8.9), the inequality (10.2) holds with

$$\beta(s) = \inf_{1 \leqslant p < 2} \Big[ C(p)^2 \, s^{\frac{q}{q-2}(1-2/p)} \Big],$$

so the right-hand side of (10.3) is bounded from above by

$$\inf_{1 \leqslant p < 2} \ \inf_{s > 0} \ \bigg[ \exp\bigg\{ -\frac{2t}{C(p)^2} \ s^{\frac{q}{q-2}(2/p-1)} \bigg\} \int |f|^2 \, d\mu + s \, \|f - \mathbf{E} f\|_q^2 \bigg].$$

In particular, we have the following assertion.

**Theorem 10.1.** *Assume that for some $a, b \geqslant 0$ and $\alpha > 0$ the rate function in the weak Poincaré type inequality* (10.1) *admits a polynomial bound*

$$C(p) \ \leqslant \ a \left( \frac{b}{2-p} \right)^{1/\alpha}, \quad 1 \leqslant p < 2. \tag{10.6}$$

*Then for any $f \in L^q(\mu)$, $q > 2$, such that $\int f \, d\mu = 0$ and for all $t \geqslant 0$*

$$\int |P_t f|^2 \, d\mu \ \leqslant \ 3 \exp\big\{ -c \, t^{\frac{\alpha}{\alpha+2}} \big\} \|f\|_q^2, \tag{10.7}$$

*where the constant $c > 0$ depends on the parameters $a$, $b$, $\alpha$, and $q$ only.*

Indeed, by Corollary 8.6, the hypothesis (10.6) implies $\beta(s) \leqslant \beta \log^{2/\alpha}(2/s)$, where $\beta = \beta_0 a^2 \, (\beta_1 b \, \frac{q}{q-2})^{2/\alpha}$ with some positive absolute constants $\beta_0$ and $\beta_1$. Hence, in order to estimate $\xi(t)$ from above, it remains to solve

$$\beta \log^{1+\frac{2}{\alpha}}(2/s) \leqslant 2t,$$

and we arrive at

$$\xi(t) \leqslant 2 \exp\Big\{ -\Big( \frac{2t}{\beta} \Big)^{\frac{\alpha}{\alpha+2}} \Big\}.$$

Finally, apply (10.5).

Now, recalling Corollary 8.3 and Remark 8.1, we obtain the hypercontractivity property (10.7) for a large family of convex probability measures.

**Corollary 10.2.** *If a probability measure $\mu$ is $\varkappa$-concave for some $\varkappa < 0$ and*

$$\int \exp\left\{ \left( \frac{|x|}{\lambda} \right)^{\alpha} \right\} d\mu(x) \leqslant 2, \quad \alpha, \lambda > 0,$$

*then it satisfies* (10.7) *for any $f \in L^q(\mu)$, $q > 2$, such that $\int f \, d\mu = 0$.*

Using a perturbation argument, one may obtain other interesting examples. In particular, they include all probability measures $\mu$ on $\mathbf{R}^n$ with densities of the form (7.1), i.e.,

$$\frac{d\mu(x)}{dx} = \frac{1}{Z}\, e^{-(a+b|x|)^\alpha}, \quad x \in \mathbf{R}^n,$$

with parameters $a \geqslant 0$, $b > 0$, and $\alpha > 0$.

At the next step, we generalize the previous results to $L^p$-spaces, so that to control the rate of convergence of $P_t f$ for norms different than $L^2$-norms.

We start with the weak Poincaré type inequality (10.2) for the functional $\Phi(f) = \|f - \mathbf{E}f\|_r$, i.e., with the family of inequalities

$$\mathrm{Var}_\mu(f) \ \leqslant\ \beta_r(s)\, \|\nabla f\|_2^2 + s\, \|f - \mathbf{E}f\|_r^2, \quad s > 0, \tag{10.8}$$

where $f$ is an arbitrary locally Lipschitz function on $\mathbf{R}^n$, $r > 2$, and $\beta_r$ is a function of the parameter $s$.

**Theorem 10.3.** *Under* (10.8), *given* $q > p > 1$ *such that* $\frac{pr}{2} = q$, *for all* $f \in L^q(\mu)$ *and* $t, s \geqslant 0$

$$\begin{aligned}\iint |P_t f(x) - P_t f(y)|^p \, d\mu(x)d\mu(y) \\ \leqslant \exp\Big\{ -\frac{4(p-1)}{p}\, \frac{t}{\beta_r(s)}\Big\} \iint |f(x) - f(y)|^p \, d\mu(x)d\mu(y) \\ + s \cdot \frac{p}{2(p-1)} \Big( \iint |f(x) - f(y)|^q \, d\mu(x)d\mu(y)\Big)^{p/q}. \end{aligned} \tag{10.9}$$

*Proof.* The argument represents a slight modification of the proof of Theorem 9.2. By (9.3), given a twice continuously differentiable, convex function $u$ on the real line and a differentiable function $v$ such that $v'^2 = u''$, we have for any $f \in \mathcal{D}$ such that $\int f\, d\mu = 0$ and for all $t, s > 0$

$$\begin{aligned}\frac{d}{dt} \int u(P_t f)\, d\mu &= -\int u''(P_t f)\, |\nabla P_t f|^2 \, d\mu \\ &= -\int |\nabla v(P_t f)|^2 \, d\mu \\ &\leqslant -\frac{1}{\beta(s)}\ \mathrm{Var}_\mu[v(P_t f)] + \frac{s}{\beta(s)}\, \|v(P_t f) - \mathbf{E}_\mu v(P_t f)\|_r^2, \end{aligned}$$

where the derivative is understood as the derivative from the right. In particular, we may take $u(z) = |z|^p$ with $p \geqslant 2$, so that $u''(z) = p(p-1)|z|^{p-2}$, and $v(z) = 2\sqrt{\frac{p-1}{p}}\ \mathrm{sign}(z)\, |z|^{p/2}$, to get

$$\frac{d}{dt}\int |P_tf|^p\,d\mu \leqslant -\frac{4(p-1)}{p}\,\frac{1}{\beta(s)}\int |P_tf|^p\,d\mu + \frac{s}{\beta(s)}\,\|\,|P_tf|^{p/2}\|_r^2 \quad (10.10)$$

provided that

$$\int \mathrm{sign}(P_tf)\,|P_tf|^{p/2}\,d\mu = 0.$$

Note that the latter holds when $P_tf$ has a distribution under $\mu$, which is symmetric about zero. A slight modification of $u(z) = |z|^p$ near zero, described in the proof of Theorem 9.2, allows one to replace the constraint $p \geqslant 2$ by the weaker condition $p > 1$. Note that, by the contraction property (9.14),

$$\|\,|P_tf|^{p/2}\|_r^2 = \|P_tf\|_{pr/2}^p = \|P_tf\|_q^p \leqslant \|f\|_q^p,$$

so (10.10) yields

$$\frac{d}{dt}\int |P_tf|^p\,d\mu \leqslant -\frac{4(p-1)}{p}\,\frac{1}{\beta_r(s)}\int |P_tf|^p\,d\mu + \frac{s}{\beta_r(s)}\,\|f\|_q^p. \quad (10.11)$$

Now, to guarantee that $P_tf$ has a symmetric distribution, we consider the product measure $\mu\otimes\mu$ on $M = \mathbf{R}^n\times\mathbf{R}^n$. With this measure we associate the semigroup $\overline{P}_t$, $t \geqslant 0$, acting on a certain space $\overline{\mathcal{D}}$ of bounded smooth functions on $\mathbf{R}^n\times\mathbf{R}^n$ with bounded partial derivatives, containing all functions of the form

$$\overline{f}(x,y) = f(x) - f(y), \quad x,y\in\mathbf{R}^n, \quad f\in\mathcal{D}.$$

It easy to see that for such functions

$$(\overline{P}_t\overline{f})(x,y) = P_tf(x) - P_tf(y), \quad (\overline{L}\overline{f})(x,y) = Lf(x) - Lf(y),$$

where $\overline{L}$ is the generator of $\overline{P}_t$.

We are going to apply (10.11) to $\overline{f}$ on the product space $(\mathbf{R}^n\times\mathbf{R}^n, \mu\otimes\mu)$, so we need a hypothesis of the form (10.8) with respect to the product measure. Note that

$$\mathrm{Var}_{\mu\otimes\mu}(\overline{f}) = 2\ \mathrm{Var}_\mu(f), \quad \mathbf{E}_{\mu\otimes\mu}|\nabla\overline{f}(x,y)|^2 = 2\mathbf{E}_\mu|\nabla f|^2$$

and, by the Jensen inequality,

$$\mathbf{E}_\mu|f-\mathbf{E}_\mu f|^r \leqslant \mathbf{E}_{\mu\otimes\mu}|\overline{f}|^r.$$

Hence (10.8) implies

$$\mathrm{Var}_{\mu\otimes\mu}(\overline{f}) \leqslant \beta_r(s)\|\nabla\overline{f}\|_2^2 + 2s\|\overline{f}\|_r^2, \quad s>0.$$

As a result, we obtain a slightly weakened form of (10.11), namely,

$$\frac{d}{dt}\int |\overline{P}_t\overline{f}|^p \, d\mu\otimes\mu \,\leqslant\, -\frac{4(p-1)}{p}\,\frac{1}{\beta_r(s)}\int |\overline{P}_t\overline{f}|^p \, d\mu\otimes\mu + \frac{2s}{\beta_r(s)}\,\|\overline{f}\|_q^p. \tag{10.12}$$

Let us note that, by virtue of the subadditivity property of the variance functional, (10.12) may be extended to the whole space $\overline{\mathcal{D}}$, however, with a worse constant in place of 2.

Thus, the function

$$\varphi(t) = \int |\overline{P}_t\overline{f}|^p \, d\mu\otimes\mu = \iint |P_tf(x) - P_tf(y)|^p \, d\mu(x)d\mu(y)$$

is continuous and has the right derivative at every point $t \geqslant 0$ satisfying the differential inequality

$$\varphi'(t) \,\leqslant\, -A\,\varphi(t) + B$$

with

$$A = \frac{4(p-1)}{\beta_r(s)p}, \quad B = \frac{2s}{\beta_r(s)}\,\|\overline{f}\|_q^p.$$

Using the change $\varphi(t) = \psi(t)e^{-At}$, we obtain

$$\varphi(t) \leqslant \varphi(0)e^{-At} + \frac{B}{A}\,(1 - e^{-At}) \leqslant \varphi(0)e^{-At} + \frac{B}{A},$$

i.e.,

$$\iint |P_tf(x) - P_tf(y)|^p \, d\mu(x)d\mu(y) \leqslant e^{-At}\iint |f(x) - f(y)|^p \, d\mu(x)d\mu(y) + \frac{B}{A}.$$

But

$$\frac{B}{A} = \frac{ps}{2(p-1)}\,\|\overline{f}\|_q^p,$$

so we arrive at the desired inequality (10.9). Finally, by continuity of $P_t$, this inequality extends from $\mathcal{D}$ to the whole space $L^q(\mu)$. □

At the expense of some constants, depending on $p$ and $q$, the inequality (10.9) may be simplified. Namely, if $\int f d\mu = 0$, the left-hand side of (10.9) majorizes $\|P_tf\|_p^p$, while the integrals on the right-hand side are bounded by $2^p\|f\|^p$ and $2^q\|f\|^q$ respectively. Hence

$$\|P_tf\|_p^p \,\leqslant\, 2^p\, e^{-\frac{4(p-1)}{p}\,\frac{t}{\beta_r(s)}}\,\|f\|_p^p + s\cdot\frac{2^p\,p}{2(p-1)}\,\|f\|_q^p.$$

**Corollary 10.4.** *Given $q > p > 1$, under (10.8) with $r = \frac{2q}{p}$, for all $f \in L^q(\mu)$ with mean zero and for all $t \geqslant 0$,*

$$\|P_t f\|_p^p \;\leqslant\; 2^p\,\|f\|_q^p\,\inf_{s>0}\Big[e^{-\frac{4(p-1)}{p}\,\frac{t}{\beta_r(s)}} + \frac{p}{2(p-1)}\,s\Big]. \tag{10.13}$$

To further simplify this bound, define the function

$$\xi(t) = \inf\Big\{s > 0 : \beta_r(s)\log\frac{2}{s} \leqslant \frac{4p}{p-1}\,t\Big\}$$

depending also on the parameters $p$ and $r$. Then the expression in the square brackets in (10.13) is bounded by

$$\frac{s}{2} + \frac{p}{2(p-1)}\,s \leqslant \frac{p}{p-1}\,s.$$

Therefore,

$$\|P_t f\|_p^p \;\leqslant\; \frac{p}{p-1}\,2^p\,\|f\|_q^p\,\xi(t). \tag{10.14}$$

Now, let us start with the weak Poincaré type inequality (10.1) with rate function $C(p)$ satisfying the bound (10.6), as in Theorem 10.1. Then, as we know from Corollary 8.6, the hypothesis (10.8) holds with

$$\beta_r(s) = \beta\log^{\frac{2}{\alpha}}\frac{2}{s},$$

where

$$\beta = 2a^2\Big(4b\,\frac{r}{r-2}\Big)^{2/\alpha}.$$

Since $r = \frac{2q}{p}$, the coefficient is

$$\beta = 2a^2\Big(4b\,\frac{q}{q-p}\Big)^{2/\alpha}.$$

We also find that

$$\xi(t) \leqslant 2\exp\Big\{-\Big(\frac{4p}{\beta(p-1)}\,t\Big)^{\alpha/(\alpha+2)}\Big\}.$$

As a result, we obtain the following generalization of Theorem 10.1.

**Theorem 10.5.** *Assume that the weak Poincaré type inequality* (10.1) *holds with rate function $C(p)$ satisfying the bound* (10.6) *with some parameters $a, b \geqslant 0$ and $\alpha > 0$. Given $q > p > 1$, for all $f \in L^q(\mu)$ with mean zero and for all $t \geqslant 0$*

$$\int |P_t f|^p\,d\mu \;\leqslant\; \frac{p}{p-1}\,2^{p+1}\,\exp\big\{-c\,t^{\frac{\alpha}{\alpha+2}}\big\}\Big(\int |f|^q\,d\mu\Big)^{p/q}, \tag{10.15}$$

*where the constant $c > 0$ depends on $a$, $b$, $\alpha$, $p$, and $q$ only.*

More precisely, we may put

$$c \;=\; \left(\frac{4p}{p-1}\right)^{\frac{\alpha}{\alpha+2}} \;\frac{\left(\frac{q-p}{q}\right)^{\frac{2}{\alpha+2}}}{(2a^2)^{\frac{2}{\alpha}}\;(4b)^{\frac{2}{\alpha+2}}}.$$

# 11 $L^2$ Decay to Equilibrium in Infinite Dimensions

## 11.1 Basic inequalities and decay to equilibrium in the product case

In this and next sections, we further simplify the notation for the expectation setting $\mu f \equiv \mathbf{E}_\mu f = \int f \, d\mu$ for the expectation of $f$ under a probability measure $\mu$. This will prove to be useful when we have to deal with more involved mathematical expressions.

Consider a probability measure on the real line of the form

$$\nu_0(dx) \equiv \frac{1}{Z} e^{-V(x)} dx$$

with $V(x) \equiv \varsigma(1+x^2)^{\frac{\alpha}{2}}$, where $0 < \alpha \leqslant 1$ and $\varsigma \in (0,\infty)$, while $Z$ denotes a normalization constant. Since $|x| \leqslant (1+x^2)^{\frac{1}{2}} \leqslant 1+|x|$, by Theorem 7.1 and Corollary 8.6, we have the following assertion.

**Lemma 11.1.** *For any $p \in (2,\infty)$ there exists $\beta \in (0,\infty)$ such that for any $s \in (0,1)$*

$$\nu_0 \left|f - \nu_0 f\right|^2 \leqslant \overline{\beta}(s)\nu_0 \left|\nabla f\right|^2 + s\left(\nu_0 \left|f - \nu_0 f\right|^p\right)^{\frac{2}{p}} \tag{11.1}$$

*with $\overline{\beta}(s) \equiv \beta\left(\log \frac{2}{s}\right)^{\frac{2}{\alpha}}$ for any function $f$, for which the right hand side is well defined.*

By a simple inductive argument, one gets the following property for corresponding product measures.

**Proposition 11.2** (product property)**.** *Suppose that $\nu_i$, $i \in \mathbb{N}$, satisfy*

$$\nu_i \left|f - \nu_i f\right|^2 \leqslant \overline{\beta}(s)\nu_i \left|\nabla_i f\right|^2 + s\left(\nu_i \left|f - \nu_i f\right|^p\right)^{\frac{2}{p}}. \tag{11.2}$$

*Then the product measure $\mu_0 \equiv \otimes_{i\in\mathbb{N}}\nu_i$ also satisfies*

$$\mu_0 |f - \mu_0 f|^2 \leqslant \overline{\beta}(s) \sum_{i \in \mathbb{N}} \mu_0 |\nabla_i f|^2 + s A_{p,\mu_0}(f) \tag{11.3}$$

*with*

$$A_{p,\mu_0}(f) \equiv \sum_{i \in \mathbb{N}} \mu_0 \left(\nu_i |f - \nu_i f|^p\right)^{\frac{2}{p}} . \tag{11.4}$$

*Proof.* Note that for $f_i \equiv \nu_i f_{i-1} \equiv \nu_{\leqslant i} f$, $i \in \mathbb{N}$, with $f_0 \equiv f$, we have

$$\mu_0 |f - \mu_0 f|^2 = \sum_{i \in \mathbb{N}} \mu_0 \nu_i |f_{i-1} - \nu_i f_{i-1}|^2 .$$

Hence, applying (11.2) to each term, we arrive at

$$\mu_0 |f - \mu_0 f|^2 \leqslant \sum_{i \in \mathbb{N}} \mu_0 \left( \overline{\beta}(s) \nu_i |\nabla_i f_{i-1}|^2 + s \left(\nu_i |f_{i-1} - \nu_i f_{i-1}|^p\right)^{\frac{2}{p}} \right) .$$

Next, we note that (by using the Minkowski and Schwartz inequalities)

$$\left(\nu_i |f_{i-1} - \nu_i f_{i-1}|^p\right)^{\frac{2}{p}} \leqslant \nu_{\leqslant i-1} \left(\nu_i |f - \nu_i f|^p\right)^{\frac{2}{p}}$$

and

$$\nu_i |\nabla_i f_{i-1}|^2 \leqslant \nu_{\leqslant i} |\nabla_i f|^2 .$$

Thus, taking into the account the fact that $\mu_0 \nu_{\leqslant i} F = \mu_0 F$ (and similarly, with $\nu_i$ in place of $\nu_{\leqslant i}$), we arrive at

$$\mu_0 |f - \mu_0 f|^2 \leqslant \overline{\beta}(s) \sum_{i \in \mathbb{N}} \mu_0 |\nabla_i f|^2 + s \sum_{i \in \mathbb{N}} \mu_0 \left(\nu_i |f - \nu_i f|^p\right)^{\frac{2}{p}} . \tag{11.5}$$

This ends the proof of the proposition. □

The Dirichlet form defines the following Markov generator:

$$L^{(0)} \equiv \sum_{i \in \mathbb{N}} L_i^{(0)},$$

with $L_i^{(0)} \equiv \varDelta_i - V'(x_i)\nabla_i$, where $\varDelta_i$ and $\nabla_i$ denote the Laplace operator and derivative with respect to the $i$th variable respectively. It is well defined on a dense domain in $L^2(\mu_0)$. As in our situation $V$ is smooth and $V'$ is bounded, the corresponding semigroup $P_t^{(0)}$ in $L^2(\mu_0)$ extends nicely to a $C_0$-semigroup onto the space of continuous functions $\mathcal{C}(\varOmega)$, where $\varOmega \equiv \mathbb{R}^{\mathbf{N}}$. Using Proposition 11.2 and the fact that functional

$$A_{p,\mu_0}(f) \equiv \sum_{i \in \mathbb{N}} \mu_0 \left(\nu_i |f - \nu_i f|^p\right)^{\frac{2}{p}}$$

is monotone with respect to the semigroup (in the sense of (10.3)), one can see that Theorems 10.1 and 10.4 hold. In particular, we have

$$\mu_0 \left| P_t^{(0)} f - \mu_0 f \right|^2 \leqslant C e^{-ct^{\frac{\alpha}{\alpha+2}}} A_{p,\mu_0}(f)$$

with some constants $C, c \in (0, \infty)$ independent of $f$.

In the rest of the paper, we prove that an inequality of a similar shape remains true for infinite systems described by nontrivial Gibbs measures. Although the corresponding functional $A_p$ may no longer be monotone, with extra work we show that the corresponding semigroups also satisfy stretched exponential decay estimate. We begin from presenting the necessary elements of the construction of the semigroups.

## *11.2 Semigroup for an infinite system with interaction.*

Let $\Omega \equiv \mathbb{R}^{\mathcal{R}}$, with a countable connected graph $\mathcal{R}$ furnished with the natural metric (given by the number of edges in the shortest path connecting two points) and with at most stretched exponential volume growth.

Let $V \equiv \varsigma(1 + x^2)^{\frac{\alpha}{2}}$, with $0 < \alpha \leqslant 1$ and $\varsigma \in (0, \infty)$. Then

$$||V^{'}||_\infty, \quad ||V^{''}||_\infty < \infty.$$

We set $V_i(\omega) \equiv V(\omega_i)$. Let $U_i(\omega) \equiv V_i(\omega) + u_i(\omega)$, where $u_i$ is a smooth function. Later on we set

$$a \equiv \sup_i \Big( 2\gamma_{ii} + \sum_{j \neq i} \gamma_{ij} \Big), \tag{11.6}$$

$$\gamma_{ij} \equiv ||\nabla_i \nabla_j u_j||_\infty \tag{11.7}$$

and assume that $a \in (0, \infty)$. We note that, by the definition of local interaction $V_i$, we automatically have $||\nabla_i^2 V_i||_\infty < \infty$, so our assumption is only about $u_j$'s. For simplicity of exposition, we assume that $\mathcal{R} = \mathbb{Z}^d$ and that the interaction is of finite range, i.e., for some $R \in (0, \infty)$ and all vertices $i$ one has $\nabla_k u_i = 0$ when dist $(i, k) \geqslant R$.

Let $P_t^\Lambda$ be a Markov semigroup associated to the generator

$$\mathcal{L}_\Lambda \equiv \sum_{i \in \mathcal{R}} L_i^{(0)} - \sum_{i \in \Lambda} \nabla_i u_i \cdot \nabla_i,$$

where

$$L_i^{(0)} \equiv \Delta_i - \nabla_i V_i(\omega) \nabla_i = \Delta_i - V'(\omega_i) \nabla_i$$

and the index $i$ indicates that derivatives are taken with respect to $\omega_i$, and $\Lambda \subset\subset \mathcal{R}$ (i.e., $\Lambda$ is a bounded subset of $\mathcal{R}$). The following lemma will play later a crucial role in the control of decay to equilibrium. Naturally, it holds for $P_t^\Lambda$ as well and is essential in defining the infinite volume semigroup as follows:

$$P_t f \equiv \lim_{\Lambda \to \mathcal{R}} P_t^\Lambda f$$

on the space of bounded continuous functions (cf., for example, [21]).

**Lemma 11.3** (finite speed of propagation of information estimate)**.** *There exist* $A, B, C \in (0, \infty)$ *such that for any smooth cylinder function* $f$ *and any* $i \in \mathcal{R}$

$$||\nabla_i P_t f||^2 \leqslant C e^{At - Bd(i,\Lambda_f)} |||f|||^2, \tag{11.8}$$

*where* $\Lambda_f \subset \mathcal{R}$ *is the smallest set* $\mathcal{O} \subset \mathcal{R}$ *such that* $f$ *depends only on* $\{\omega_i : i \in \mathcal{O}\}$ *and*

$$|||f|||^2 \equiv \sum_{i \in \mathcal{R}} ||\nabla_i f||_\infty^2.$$

The proof is based on the following arguments (note that, under our smoothness assumptions on the interaction, the pointwise operations are well justified):

$$\begin{aligned}
\frac{d}{d\tau} P_\tau |\nabla_i P_{t-\tau} f|^2 &= P_\tau \Big( \mathcal{L} |\nabla_i P_{t-\tau} f|^2 - 2 \nabla_i P_{t-\tau} f \cdot \mathcal{L} \nabla_i P_{t-\tau} f \Big) \\
&\quad + 2 P_\tau \left( \nabla_i P_{t-\tau} f \cdot [\mathcal{L}, \nabla_i] P_{t-\tau} f \right) \\
&\geqslant 2 P_\tau \Big( \nabla_i P_{t-\tau} f \cdot [\mathcal{L}, \nabla_i] P_{t-\tau} f \Big) \\
&= P_\tau \Big( -2 \nabla_i^2 U_i |\nabla_i P_{t-\tau} f|^2 - 2 \sum_{j \neq i} \nabla_i \nabla_j u_j \nabla_i P_{t-\tau} f \nabla_j P_{t-\tau} f \Big) \\
&\geqslant -(2 ||\nabla_i^2 U_i||_\infty + \sum_{j \neq i} ||\nabla_i \nabla_j u_j||_\infty) \cdot P_\tau |\nabla_j P_{t-\tau} f|^2 \\
&\quad - \sum_{j \neq i} ||\nabla_i \nabla_j u_j||_\infty P_\tau |\nabla_j P_{t-\tau} f|^2 .
\end{aligned}$$

Hence, with the notation introduced in (11.6) before the lemma, we have

$$\|\nabla_i P_t f\|^2 \leqslant e^{at} \|\nabla_i f\|^2 + \sum_{j \neq i} \gamma_{ij} \int_0^t e^{a(t-\tau)} \|\nabla_j P_\tau f\|^2 \, d\tau.$$

In particular, if $i \notin \Lambda_f$, we get

$$\|\nabla_i P_t f\|^2 \leqslant \sum_{j \neq i} \gamma_{ij} \int_0^t e^{a(t-\tau)} \|\nabla_j P_\tau f\|^2 \, d\tau.$$

By standard arguments (cf. [21] and the references therein), this leads to the desired estimate of final speed of propagation of information (11.8).

## *11.3* $L^2$ *decay*

Our way to study the $L^2$ decay of the semigroup is as follows. Suppose that $\mu$ satisfies $\mu E_i f = \mu f$ for any $i \in \mathcal{R}$ with the following probability kernels:

$$E_i(f) \equiv E_i^\omega(f) \equiv \delta_\omega \left( \frac{\displaystyle\int f e^{-U_i} d\omega_i}{\displaystyle\int e^{-U_i} d\omega_i} \right) = \delta_\omega \left( \frac{\displaystyle\int f e^{-u_i} d\nu_i}{\displaystyle\int e^{-u_i} d\nu_i} \right), \tag{11.9}$$

where $\delta_\omega$ denotes the Dirac mass concentrated at $\omega$ and, by definition, $\nu_i$ is an isomorphic copy of the probability measure $\nu_0$. Then $P_t$ is a symmetric semigroups in $L^2(\mu)$ with quadratic form of the generator given by

$$\mu|\nabla f|^2 \equiv \sum_{i \in \mathcal{R}} \mu|\nabla_i f|^2.$$

Let

$$A_p(f) \equiv A_{p,\mu}(f) \equiv \sum_{i \in \mathcal{R}} \mu \left( \nu_i \left| f - \nu_i f \right|^p \right)^{\frac{2}{p}}.$$

With this notation, we have the following assertion.

**Lemma 11.4.** *Assume that, with a positive function* $\beta(s)$*,*

$$\mu(f - \mu f)^2 \leqslant \beta(s)\ \mu|\nabla f|^2 + s A_p(f).$$

*Then*

$$\mu \left( P_t f - \mu f \right)^2 \leqslant \inf_s \left\{ e^{-\frac{t}{\beta(s)}} \mu \left( f - \mu f \right)^2 + s \sup_{0 \leqslant \tau \leqslant t} A_p(P_\tau f) \right\}.$$

The above follows from the following simple arguments (cf., for example, [35] and the references therein) similar to those in Section 10. For $f_t \equiv P_t f$ we have

$$\frac{d}{dt} \mu \left( f_t - \mu f \right)^2 = -2\mu|\nabla f_t|^2 \leqslant -\frac{2}{\beta(s)} \mu \left( f_t - \mu f \right)^2 + \frac{2s}{\beta(s)} A_p(f_t).$$

Hence

$$\mu\left(f_t - \mu f\right)^2 \leqslant e^{-\frac{2t}{\beta(s)}} \mu\left(f - \mu f\right)^2 + \int_0^t e^{-\frac{2(t-\tau)}{\beta(s)}} \frac{2s}{\beta(s)} A_p(f_\tau) d\tau$$

$$\leqslant e^{-\frac{2t}{\beta(s)}} \mu\left(f - \mu f\right)^2 + s \sup_{0 \leqslant \tau \leqslant t} A_p(f_\tau).$$

To go the route based on Lemma 11.4, we need an estimate for the functional $A_p$.

**Proposition 11.5** (estimate of $A_p(f_\tau)$)**.** *Suppose that* $\Lambda \equiv \Lambda_t \subset\subset \mathcal{R}$ *satisfies*

$$\mathrm{dist}\left(\Lambda^c, \Lambda_f\right) \geqslant \frac{1}{4} \mathrm{diam}\left(\Lambda\right)$$

*with* $\mathrm{diam}\left(\Lambda\right) = 16 \frac{A}{B} t$. *Then*

$$\sup_{0 \leqslant \tau \leqslant t} A_p(f_\tau) \leqslant |\Lambda_t| \cdot G\left(\mu\left|f - \mu f\right|^p\right)^{\frac{2}{p}} + D e^{-A\ t} \cdot |||f|||^2$$

*with some constants* $D, G \in (0, \infty)$ *independent of* $f$.

*Proof.* For any $\Lambda \subset\subset \mathcal{R}$ we have

$$A_p(f_\tau) \equiv \sum_{i \in \mathcal{R}} \mu\left(\nu_i \left|f_\tau - \nu_i f_\tau\right|^p\right)^{\frac{2}{p}}$$

$$= \sum_{i \in \Lambda} \mu\left(\nu_i \left|f_\tau - \nu_i f_\tau\right|^p\right)^{\frac{2}{p}} + \sum_{i \in \Lambda^c} \mu\left(\nu_i \left|f_\tau - \nu_i f_\tau\right|^p\right)^{\frac{2}{p}}.$$

Since for $p > 2$ we have

$$\mu\left(\left(\nu_i \left|f_\tau - \nu_i f_\tau\right|^p\right)^{\frac{2}{p}}\right) \leqslant 4\mu\left(\left(\nu_i \left|f_\tau - \mu f_\tau\right|^p\right)^{\frac{2}{p}}\right)$$

$$\leqslant 4 e^{\frac{4}{p} \sup_i ||u_i||_\infty} \mu\left(\left(E_i \left|f_\tau - \mu f_\tau\right|^p\right)^{\frac{2}{p}}\right)$$

$$\leqslant 4 e^{\frac{4}{p} \sup_i ||u_i||_\infty} \left(\mu\left|f_\tau - \mu f_\tau\right|^p\right)^{\frac{2}{p}}$$

$$\leqslant 4 e^{\frac{4}{p} \sup_i ||u_i||_\infty} \left(\mu\left|f - \mu f\right|^p\right)^{\frac{2}{p}},$$

where we used the triangle and Hölder inequalities and gained a factor $e^{\frac{4}{p} \sup_i ||u_i||_\infty}$ while passing from expectations with the measure $\nu_i$ to the expectation with the conditional expectation $E_i$. Thus,

$$A_p(f_\tau) \leqslant |\Lambda| \cdot G\left(\mu\left|f - \mu f\right|^p\right)^{\frac{2}{p}} + \sum_{i \in \Lambda^c} \mu\left(\nu_i \left|f_t - \nu_i f_t\right|^p\right)^{\frac{2}{p}}$$

with the constant $G \equiv 4 e^{\frac{4}{p} \sup_i ||u_i||_\infty}$. To estimate the sum over $i \in \Lambda^c$, we note that for $\omega, \widetilde{\omega} \in \Omega$ satisfying $\omega_j = \widetilde{\omega}_j$ for $j \neq i$

$$|f_\tau(\omega) - f_\tau(\widetilde{\omega})| = \Big| \int_{\widetilde{\omega}_i}^{\omega_i} dx\ \nabla_i f_\tau(x \bullet \omega_{\mathcal{R}\setminus i}) \Big| \leqslant |\omega_i - \widetilde{\omega}_i| \cdot \|\nabla_i f_\tau\|_\infty$$

with configuration

$$[x \bullet \omega_{\mathcal{R}\setminus i}]_j \equiv \delta_{ij} x + (1 - \delta_{ij})\omega_j.$$

Thus, we get

$$|f_\tau(\omega) - f_\tau(\widetilde{\omega})| \leqslant |\omega_i - \widetilde{\omega}_i| \cdot C^{\frac{1}{2}} e^{\frac{A}{2}s - \frac{B}{2}d(i,\Lambda_f)} |||f|||$$

which implies

$$\mu\left(\nu_i\, |f_\tau - \nu_i f_\tau|^p\right)^{\frac{2}{p}} \leqslant 4Ce^{A\tau - Bd(i,\Lambda_f)} \left(\nu_0 |\omega_0|^p\right)^{\frac{2}{p}} \cdot |||f|||^2.$$

Since

$$\nu_0(dx) = \frac{1}{Z} e^{-\varsigma(1+x^2)^{\frac{\alpha}{2}}} dx,$$

one can obtain the following estimate (using the Stirling bound):

$$\left(\nu_0|\omega_0|^p\right)^{\frac{2}{p}} \leqslant C' e^{\frac{4}{\alpha}\log p}$$

with some constant $C' \equiv C'(\alpha, \varsigma) \in (0, \infty)$ independent of $p \in (2, \infty)$. (This is an important place where we take advantage of oscillations in $L^p$; would we have the functional Osc as in (1.6), we would be in trouble.) Hence we obtain the following bound:

$$\sum_{i \in \Lambda^c} \mu\left(\nu_i\, |f_\tau - \nu_i f_\tau|^p\right)^{\frac{2}{p}} \leqslant De^{A\tau - \frac{B}{2}d(\Lambda^c, \Lambda_f)} \cdot |||f|||^2$$

with

$$D \equiv 4CC' e^{\frac{4}{\alpha}\log p} \sum_{i \in \mathcal{R}:\mathrm{dist}\,(i,\Lambda_f) \geqslant d(\Lambda^c, \Lambda_f)+1} e^{-\frac{B}{2}d(i,\Lambda_f)}$$

with the series being convergent due to our assumption about slower than exponential volume growth of $\mathcal{R}$. For $\tau \in [0, t]$, choosing $\Lambda \equiv \Lambda_t$ such that $\mathrm{dist}\,(\Lambda^c, \Lambda_f) \geqslant \frac{1}{4}\,\mathrm{diam}\,(\Lambda)$ with $\mathrm{diam}\,(\Lambda) = 16\frac{A}{B}t$, we get

$$\sum_{i \in \Lambda^c} \mu\left(\nu_i\, |f_\tau - \nu_i f_\tau|^p\right)^{\frac{2}{p}} \leqslant De^{-A\ t} \cdot |||f|||^2.$$

Combining all the above, we arrive at the following estimate:

$$\sup_{0 \leqslant \tau \leqslant t} A_p(f_\tau) \leqslant |\Lambda_t| \cdot G\left(\mu\, |f - \mu f|^p\right)^{\frac{2}{p}} + De^{-A\ t} \cdot |||f|||^2.$$

The proposition is proved. □

Given the above estimate for $A_p(f_t)$, we conclude with the following result.

**Theorem 11.6.** *Let $\Lambda \equiv \Lambda_t$ be an increasing family of bounded subsets of $\mathcal{R}$ such that* $\text{dist}\,(\Lambda^c, \Lambda_f) \geqslant \frac{1}{4}\,\text{diam}\,(\Lambda)$, *with* $\text{diam}\,(\Lambda) = 16\frac{A}{B}t$ *and* $|\Lambda_t| \leqslant e^{\text{diam}\,(\Lambda_t)^\theta}$, *with some $\theta \in (0,1)$ for all sufficiently large $\Lambda_t$. Assume that for a positive function $\beta(s) = \xi^{-1}\left(\log(1/s)\right)^\eta$ defined with some $\xi, \eta \in (0,\infty)$*

$$\mu(f-\mu f)^2 \leqslant \beta(s)\ \mu|\nabla f|^2 + sA_p(f)$$

*for each $s \in (0,1)$.*

*If $\theta \in (0, 1/\eta)$, then there exist constant $\zeta, J \in (0,\infty)$, and $\varepsilon \in (0,1)$ such that*

$$\mu(f_t-\mu f_t)^2 \leqslant e^{-\zeta t^\varepsilon}\ J\left(\mu\left(f-\mu f\right)^2 + \left(\mu\left|f-\mu f\right|^p\right)^{\frac{2}{p}} + |||f|||^2\right).$$

Remark 11.1. In the case of a regular lattice $\mathbb{Z}^d$ and finite range interactions, one would have $|\Lambda_t| \sim t^d$. Our weaker growth assumption allows one to include more general graphs, as well as interactions which are not of finite range.

We note that for our considerations it is relevant only what is the behavior of $\beta(s)$ for small $s$. This determines the long time behavior (while the short time estimates can be compensated by a choice of constant $J$). This allows us to disregard factor 2 (or any similar numerical factor) from within the log in $\beta(s)$ as compared to estimates used in the product case.

*Proof of Theorem 11.6.* By Lemma 11.4 and Proposition 11.5, we have

$$\begin{aligned}&\mu(f_t-\mu f_t)^2\\ &\leqslant \inf_s\left\{e^{-\frac{t}{\beta(s)}}\mu\left(f-\mu f\right)^2 + s\left(|\Lambda_t|\cdot G\left(\mu\left|f-\mu f\right|^p\right)^{\frac{2}{p}} + De^{-A\ t}\cdot|||f|||^2\right)\right\}.\end{aligned}$$

Hence, choosing $s = e^{-t^\sigma}$ with $\sigma \in (\theta, 1/\eta)$, we obtain

$$\begin{aligned}\mu(f_t-\mu f_t)^2 &\leqslant \exp\{-\xi t^{1-\sigma\eta}\}\mu\left(f-\mu f\right)^2\\ &\quad + e^{-t^\sigma}\left(e^{(16\frac{A}{B}t)^\theta}\cdot G\left(\mu\left|f-\mu f\right|^p\right)^{\frac{2}{p}} + De^{-A\ t}e^{\frac{4}{\alpha}\log p}\cdot|||f|||^2\right).\end{aligned}$$

Thus, if $\sigma \in (\theta, 1/\eta)$, then there exists a constant $J \in (0,\infty)$ such that with $\varepsilon \equiv \min(1-\sigma\eta,\ \sigma-\theta)$ and any $\zeta \in (0, \min(1,\xi))$ we have

$$\mu(f_t-\mu f_t)^2 \leqslant e^{-\zeta t^\varepsilon}\ J\left(\mu\left(f-\mu f\right)^2 + \left(\mu\left|f-\mu f\right|^p\right)^{\frac{2}{p}} + |||f|||^2\right).$$

The theorem is proved. □

## 12 Weak Poincaré Inequalities for Gibbs Measures

In this section, we prove a weak Poincaré inequality for Gibbs measures with slowly decaying tails in the region of strong mixing property. Using this result, we obtain an estimate for the decay to equilibrium in $L_2$ for all Lipschitz cylinder functions with the same stretched exponential rate.

For $\Lambda \subset\subset \mathcal{R}$ we define the following conditional expectations (generalizing the $E_i$ introduced in (11.9))

$$dE_\Lambda \equiv \delta_\omega \left( \frac{\int f e^{-u_\Lambda} d\nu_\Lambda}{\int e^{-u_\Lambda} d\nu_\Lambda} \right)$$

with some smooth function $u_\Lambda$ and $\nu_\Lambda \equiv \otimes_{i\in\Lambda}\nu_i$, so that for $\Lambda_0 \subset \Lambda$ we have

$$dE_{\Lambda|\Sigma_{\Lambda_0}} \equiv \rho_{\Lambda_0} d\nu_{\Lambda_0}$$

with $|| \log \rho_{\Lambda_0}||_\infty \leqslant \phi|\Lambda_0|$ with some numerical constant $\phi \in (0,\infty)$. Recall that, by definition, a Gibbs measure satisfies

$$\mu E_\Lambda(f) = \mu f$$

for each finite $\Lambda$ and any integrable function $f$ (cf., for example, [21]).

We begin from the following lemma.

**Lemma 12.1** (perturbation lemma)**.** *Suppose that $\nu_i$, $i \in \mathbb{N}$, satisfy*

$$\nu_i \left|f - \nu_i f\right|^2 \leqslant \overline{\beta}(s)\nu_i \left|\nabla_i f\right|^2 + s\left(\nu_i \left|f-\nu_i f\right|^p\right)^{\frac{2}{p}}$$

*with a function $\overline{\beta} : (0, s_0) \to \mathbb{R}^+$, for some $s_0 > 0$. Then the conditional expectation $E_i \equiv \frac{1}{Z_i} e^{-u_i} d\nu_i$ satisfies*

$$E_i \left|f - E_i f\right|^2 \leqslant \widetilde{\beta}(s)E_i \left|\nabla_i f\right|^2 + s\left(\nu_i \left|f-\nu_i f\right|^p\right)^{\frac{2}{p}}$$

*with*

$$\widetilde{\beta}(s) \equiv e^{\operatorname{osc}(u_i)}\overline{\beta}(se^{-\operatorname{osc}(u_i)})$$

*for $s \in (0, s_0 e^{\operatorname{osc} u_i})$, where $\operatorname{osc}(u_i) \equiv \sup u_i - \inf u_i$.*

*If $f$ depends on $\omega_\Gamma$, with $\Gamma \cap \Lambda \equiv \Lambda_0$, then*

$$E_\Lambda \left|f - E_\Lambda f\right|^2 \leqslant \widetilde{\beta}_\Lambda(s)E_\Lambda \left|\nabla_\Lambda f\right|^2 + se^{2\phi|\Lambda_0|}\sum_{i\in\Lambda_0} E_\Lambda\left(\nu_i \left|f-\nu_i f\right|^p\right)^{\frac{2}{p}}$$

*with $\widetilde{\beta}_\Lambda(s) \equiv e^{2\phi|\Lambda_0|}\overline{\beta}(e^{-2\phi|\Lambda_0|}s)$ for $s \in (0, s_0 e^{2\phi|\Lambda_0|})$.*

*Proof.* We have

$$E_i |f - E_i f|^2 \leqslant E_i |f - \nu_i f|^2 = \int |f - \nu_i f|^2 \frac{1}{Z_i} e^{-u_i} d\nu_i \\ \leqslant \frac{1}{Z_i} e^{-\inf u_i} \nu_i |f - \nu_i f|^2 .$$

Hence, by the assumed inequality for $\nu_i$, we get

$$E_i |f - E_i f|^2 \leqslant \frac{1}{Z_i} e^{-\inf u_i} \overline{\beta}(s) \nu_i |\nabla_i f|^2 + s \frac{1}{Z_i} e^{-\inf u_i} \left(\nu_i |f - \nu_i f|^p\right)^{\frac{2}{p}} \\ \leqslant e^{\sup u_i - \inf u_i} \overline{\beta}(s) E_i |\nabla_i f|^2 + s e^{\sup u_i - \inf u_i} \left(\nu_i |f - \nu_i f|^p\right)^{\frac{2}{p}} .$$

Hence

$$E_i |f - E_i f|^2 \leqslant \widetilde{\beta}(s) E_i |\nabla_i f|^2 + s \left(\nu_i |f - \nu_i f|^p\right)^{\frac{2}{p}}$$

with

$$\widetilde{\beta}(s) \equiv e^{\mathrm{osc}\ u_i} \overline{\beta}(s e^{-\mathrm{osc}\ u_i})$$

for $s \in (0, s_0 e^{\mathrm{osc}\ u_i})$, where $\mathrm{osc}\ u_i \equiv \sup u_i - \inf u_i$. Similarly,

$$E_\Lambda |f - E_\Lambda f|^2 \leqslant E_\Lambda |f - \nu_\Lambda f|^2 = \int |f - \nu_\Lambda f|^2 \frac{1}{Z_\Lambda} e^{-u_\Lambda} d\nu_\Lambda \\ \leqslant \frac{1}{Z_\Lambda} e^{-\inf u_\Lambda} \nu_\Lambda |f - \nu_\Lambda f|^2$$

and therefore (using the product property of Weak Poincaré inequality as in Proposition 11.2),

$$E_\Lambda |f - E_\Lambda f|^2 \leqslant \frac{1}{Z_\Lambda} e^{-\inf u_\Lambda} \overline{\beta}(s) \nu_\Lambda |\nabla_\Lambda f|^2 \\ + s \frac{1}{Z_\Lambda} e^{-\inf u_\Lambda} \sum_{i \in \Lambda} \nu_\Lambda \left(\nu_i |f - \nu_i f|^p\right)^{\frac{2}{p}} \\ \leqslant e^{\mathrm{osc}\,(u_\Lambda)} \overline{\beta}(s) E_\Lambda |\nabla_\Lambda f|^2 + s e^{\mathrm{osc}\,(u_\Lambda)} \sum_{i \in \Lambda} E_\Lambda \left(\nu_i |f - \nu_i f|^p\right)^{\frac{2}{p}}$$

with $\mathrm{osc}\,(u_\Lambda) \equiv \sup u_\Lambda - \inf u_\Lambda$. In the case where $f$ depends on $\omega_\Gamma$, with $\Gamma \cap \Lambda \equiv \Lambda_0$, one can stream-line the above arguments as follows. Noting that $dE_{\Lambda|\Sigma_{\Lambda_0}} \equiv \rho_{\Lambda_0} d\nu_{\Lambda_0}$ with $||\log \rho_{\Lambda_0}||_\infty \leqslant \phi|\Lambda_0|$ with some numerical constant $\phi \in (0, \infty)$, by similar arguments as above, we obtain

$$E_\Lambda \big|f - E_\Lambda f\big|^2 = E_{\Lambda|\Sigma_{\Lambda_0}} \big|f - E_{\Lambda|\Sigma_{\Lambda_0}} f\big|^2 \leqslant E_{\Lambda|\Sigma_{\Lambda_0}} \big|f - \nu_{\Lambda_0} f\big|^2 \\ \leqslant e^{2\phi|\Lambda_0|} \overline{\beta}(s) E_\Lambda \big|\nabla_\Lambda f\big|^2 + s e^{2\phi|\Lambda_0|} \sum_{i \in \Lambda_0} E_\Lambda \big(\nu_i \big|f - \nu_i f\big|^p\big)^{\frac{2}{p}} .$$

The lemma is proved. □

Later on we consider a given set $\Lambda \subset\subset \mathcal{R}$ (for example, a ball of radius $L \in \mathbb{N}$ in a suitable metric of the graph) and write $\Lambda + j$ to denote a similar set around a point $j \in \mathcal{R}$. (If the graph $\mathcal{R}$ admits a structure of a linear space, this will coincide with a translation of $\Lambda$ by the vector $j$.)

**Lemma 12.2** (product property bis)**.** *Suppose that*

$$E_\Lambda \left| f - E_\Lambda f \right|^2 \leqslant \widetilde{\beta}_\Lambda(s) E_\Lambda \left| \nabla_\Lambda f \right|^2 + s \sum_{i \in \Lambda_0} E_\Lambda \left( \nu_i \left| f - \nu_i f \right|^p \right)^{\frac{2}{p}} .$$

*Let* $\Gamma \equiv \bigcup_{l \in \mathbb{N}} \Lambda + j_l$ *with* $j_l$ *such that* $\operatorname{dist}(\Lambda + j_l, \Lambda + j_{l'}) \geqslant 2R$*, for* $l \neq l'$*. Assume that* $E_\Lambda$ *satisfy the following local Markov property:*

$$\forall f \in \Sigma_\Lambda \quad \Longrightarrow \quad E_\Lambda(f) \in \Sigma_{\Lambda_R},$$

*where* $\Lambda_R \equiv \{ j \in \mathcal{R} : \operatorname{dist}(j, \Lambda) \leqslant R \}$ *for a given* $R \geqslant 1$*. Then*

$$E_\Gamma \left| f - E_\Gamma f \right|^2 \leqslant \widetilde{\beta}_\Lambda(s) E_\Gamma |\nabla_\Gamma f|^2 + s \sum_{i \in \Gamma} E_\Gamma \left( \nu_i \left| f - \nu_i f \right|^p \right)^{\frac{2}{p}} .$$

*If* $f \in \Sigma_\Theta$ *(i.e.,* $\Lambda_f \subseteq \Theta$*) and* $\Lambda_f \cap \Gamma \subset \bigcup_l \Lambda_0 + j_l$*, then the above inequality holds with* $\widetilde{\beta}(s) \equiv \widetilde{\beta}_{\Lambda_0}(s)$*.*

Remark 12.1. The local Markov property is true when the interaction is of finite range $R$.

Because of the local Markov property, in our setup $E_\Gamma$ acts as a product measure. Therefore, the proof is similar to the proof of Proposition 11.2 (product property).

Later on we consider a family of $\Gamma_k \subset \mathcal{R}$, $k \in \mathbb{N}$. Let $\boldsymbol{\Pi}_n(f) \equiv E_{\Gamma_n} \ldots E_{\Gamma_1}(f)$. We note that, as in [38, 39], setting

$$f_0 \equiv f \quad \text{and} \quad f_n \equiv E_{\Gamma_n} f_{n-1} = \boldsymbol{\Pi}_n(f),$$

we have

$$\mu \left( f - \mu f \right)^2 = \sum_{n \in \mathbb{N}} \mu \; E_{\Gamma_n} \left( f_{n-1} - E_{\Gamma_n} f_{n-1} \right)^2 .$$

Hence, by Lemma 12.2, we get

$$\begin{aligned} E_{\Gamma_n} \left( f_{n-1} - E_{\Gamma_n} f_{n-1} \right)^2 \leqslant \widetilde{\beta}_\Lambda(s) E_{\Gamma_n} |\nabla_{\Gamma_n} f_{n-1}|^2 \\ + s \sum_{i \in \Gamma_n} E_{\Gamma_n} \left( \nu_i \left| f_{n-1} - \nu_i f_{n-1} \right|^p \right)^{\frac{2}{p}} . \end{aligned}$$

Now, we prove the following bound for expectation of terms involving the $p$th norms.

**Lemma 12.3.**

$$\sum_{i\in\Gamma_{n+1}} \mu\big(\nu_i\big|E_{\Gamma_n}F-\nu_iE_{\Gamma_n}F\big|^p\big)^{\frac{2}{p}} \leqslant \sum_{i\in\Gamma_{n+1},\ j} \eta_{ij}\mu\big(\nu_j\big|F-\nu_jF\big|^p\big)^{\frac{2}{p}},$$

*where* $\eta_{ii}\equiv 2e^{6||u_i||}$ *and*

$$\eta_{ij}\equiv 2\sum_{k(i):j\in\Lambda_{k(i)}} \left[\operatorname{osc}_{\Lambda_{k(i)}\cap\Lambda_F}\left(E_{\Lambda_{k(i)}\setminus\Lambda_F}(D_i)\right)\right]^2\cdot e^{4\phi(|\Lambda_{k(i)}\cap\Lambda_F|+\frac{1}{2})}|\Lambda_{k(i)}\cap\Lambda_F|$$

*with*

$$D_i\equiv\frac{\rho_{\Lambda_{k(i)}}(\omega_{\Lambda_{k(i)}}\bullet\omega_i\bullet\omega_{\mathcal{R}\setminus\Lambda_{k(i)}\cup\{i\}})}{\rho_{\Lambda_{k(i)}}(\omega_{\Lambda_{k(i)}}\bullet\widetilde{\omega}_i\bullet\omega_{\mathcal{R}\setminus\Lambda_{k(i)}\cup\{i\}})}-1,$$

*where* $\Lambda_{k(i)}\subset\Gamma_n$, $i\in\Gamma_{n+1}$, *is such that* $i\in\partial_R\Lambda_{k(i)}\equiv\{j\in\Lambda^c_{k(i)}:\operatorname{dist}(j,\Lambda_{k(i)})\leqslant R\}$.

*Proof.* First we note that for $i\in\Gamma_n\cap\Gamma_{n+1}$ the quantity $E_{\Gamma_n}F-\nu_iE_{\Gamma_n}F$ vanishes. For $i\in\Gamma_{n+1}$ let $\Lambda_{k(i)}\subset\Gamma_n$ be such that

$$i\in\partial_R\Lambda_{k(i)}\equiv\{j\in\Lambda^c_{k(i)}:\operatorname{dist}(j,\Lambda_{k(i)})\leqslant R\}.$$

Let $\widetilde{\Gamma}_n^{(i)}\equiv\Gamma_n\setminus\widetilde{\Lambda}^{(i)}$ and $\widetilde{\Lambda}^{(i)}\equiv\cup\Lambda_{k(i)}$. Note that $E_{\Gamma_n}F=E_{\widetilde{\Gamma}_n^{(i)}}E_{\widetilde{\Lambda}^{(i)}}F$ and $\nu_iE_{\widetilde{\Gamma}_n^{(i)}}=E_{\widetilde{\Gamma}_n^{(i)}}\nu_i$. Hence, using the Minkowski inequality for the $L_p(\nu_i)$ norm and the Schwartz inequality for $E_{\widetilde{\Gamma}_n^{(i)}}$, we get

$$\big(\nu_i\big|E_{\widetilde{\Gamma}_n^{(i)}}E_{\widetilde{\Lambda}^{(i)}}F-\nu_iE_{\widetilde{\Gamma}_n^{(i)}}E_{\widetilde{\Lambda}^{(i)}}F\big|^p\big)^{\frac{2}{p}}\leqslant E_{\widetilde{\Gamma}_n^{(i)}}\big(\nu_i\big|E_{\widetilde{\Lambda}^{(i)}}F-\nu_iE_{\widetilde{\Lambda}^{(i)}}F\big|^p\big)^{\frac{2}{p}}.$$

On the other hand, we have

$$\begin{aligned}\left(\nu_i\left|E_{\widetilde{\Lambda}^{(i)}}F-\nu_iE_{\widetilde{\Lambda}^{(i)}}F\right|^p\right)^{\frac{2}{p}}&\leqslant 2\left(\nu_i\left|E_{\widetilde{\Lambda}^{(i)}}(F-\nu_iF)\right|^p\right)^{\frac{2}{p}}\\&\quad+2\left(\nu_i\left|[E_{\widetilde{\Lambda}^{(i)}},\nu_i]F\right|^p\right)^{\frac{2}{p}},\end{aligned}\tag{12.1}$$

where

$$[E_{\widetilde{\Lambda}^{(i)}},\nu_i]F\equiv E_{\widetilde{\Lambda}^{(i)}}\nu_iF-\nu_iE_{\widetilde{\Lambda}^{(i)}}F.$$

The first term on the right-hand side of (12.1) can be bounded as follows:

$$\begin{aligned}2\big(\nu_i\big|E_{\widetilde{\Lambda}^{(i)}}(F-\nu_iF)\big|^p\big)^{\frac{2}{p}}&\leqslant 2e^{4||u_i||}\big(\nu_i\big|E'_{\widetilde{\Lambda}^{(i)}}|F-\nu_iF\big|^p\big)^{\frac{2}{p}}\\&\leqslant 2e^{6||u_i||}E_{\widetilde{\Lambda}^{(i)}}\left(\nu_i|F-\nu_iF|^p\right)^{\frac{2}{p}},\end{aligned}\tag{12.2}$$

where $E'_{\widetilde{\Lambda}^{(i)}}$ denotes an expectation with interaction $u_i$ removed so it commutes with $\nu_i$ expectation, and we can apply the Minkowski inequality (for the $L_p(\nu_i)$ norm) and the Schwartz inequality for $E'_{\widetilde{\Lambda}^{(i)}}$ at the end inserting back the interaction $u_i$.

The second term on the right-hand side of (12.1) is estimated as follows. First we note that

$$2\big(\nu_i\big|[E_{\widetilde{\Lambda}^{(i)}},\nu_i]F\big|^p\big)^{\frac{2}{p}} \leqslant 2\sum \big(\nu_i\big|[E_{\Lambda_{k(i)}},\nu_i]F\big|^p\big)^{\frac{2}{p}}. \tag{12.3}$$

Next, we observe that

$$[E_{\Lambda_{k(i)}},\nu_i]F = \int \nu_i(d\widetilde{\omega}_i)\big\{E_{\Lambda_{k(i)}}\big(D_i(F - E_{\Lambda_{k(i)}}F)\big)\big\}, \tag{12.4}$$

where

$$D_i \equiv \frac{\rho_{\Lambda_{k(i)}}\big(\omega_{\Lambda_{k(i)}} \bullet \omega_i \bullet \omega_{\mathcal{R}\setminus\Lambda_{k(i)}\cup\{i\}}\big)}{\rho_{\Lambda_{k(i)}}\big(\omega_{\Lambda_{k(i)}} \bullet \widetilde{\omega}_i \bullet \omega_{\mathcal{R}\setminus\Lambda_{k(i)}\cup\{i\}}\big)} - 1.$$

If $F$ depends on variables $\Lambda_{k(i)} \cap \Lambda_F$, then

$$\begin{aligned}
&\big|E_{\Lambda_{k(i)}}\big(D_i(F - E_{\Lambda_{k(i)}}F)\big)\big| \\
&\qquad = \big|E_{\Lambda_{k(i)}}\big(E_{\Lambda_{k(i)}\setminus\Lambda_F}(D_i)(F - E_{\Lambda_{k(i)}}F)\big)\big| \\
&\qquad \leqslant \operatorname{osc}\,\big(E_{\Lambda_{k(i)}\setminus\Lambda_F}(D_i)\big)\cdot E_{\Lambda_{k(i)}}\big|F - E_{\Lambda_{k(i)}}F\big|
\end{aligned} \tag{12.5}$$

with oscillation over variables indexed by points in $\Lambda_{k(i)} \setminus \Lambda_F$. Thus,

$$\begin{aligned}
&\big(\nu_i\big|[E_{\Lambda_{k(i)}},\nu_i]F\big|^p\big)^{\frac{2}{p}} \\
&\qquad \leqslant \big[\operatorname{osc}\,\big(E_{\Lambda_{k(i)}\setminus\Lambda_F}(D_i)\big)\big]^2\cdot \nu_i E_{\Lambda_{k(i)}}\big|F - E_{\Lambda_{k(i)}}F\big|^2.
\end{aligned} \tag{12.6}$$

Using (12.3)–(12.6), we arrive at

$$\begin{aligned}
&2\big(\nu_i\big|[E_{\widetilde{\Lambda}^{(i)}},\nu_i]F\big|^p\big)^{\frac{2}{p}} \\
&\qquad \leqslant 2\sum\big[\operatorname{osc}\,\big(E_{\Lambda_{k(i)}\setminus\Lambda_F}(D_i)\big)\big]^2\cdot \nu_i E_{\Lambda_{k(i)}}\big|F - E_{\Lambda_{k(i)}}F\big|^2.
\end{aligned} \tag{12.7}$$

Next, we note that

$$E_{\Lambda_{k(i)}}\big|F - E_{\Lambda_{k(i)}}F\big|^2 \leqslant e^{2\phi|\Lambda_{k(i)}\cap\Lambda_F|}\nu_{\Lambda_{k(i)}\cap\Lambda_F}\big|F - \nu_{\Lambda_{k(i)}\cap\Lambda_F}F\big|^2.$$

On the other hand, choosing a lexicographic order $\{j_l \in \Lambda\}_{l=1\ldots|\Lambda|}$, we have

$$\begin{aligned}
\nu_\Lambda|F - \nu_\Lambda F|^2 &= \nu_\Lambda\Big|\sum_{l=1\ldots|\Lambda|-1}\nu_{\Lambda_l}F - \nu_{\Lambda_{l+1}}F\Big|^2 \\
&\leqslant |\Lambda|\sum_{j\in\Lambda}\nu_\Lambda\nu_j|F - \nu_jF|^2 \leqslant |\Lambda|\sum_{j\in\Lambda}\nu_\Lambda\left(\nu_j|F - \nu_jF|^p\right)^{\frac{2}{p}}
\end{aligned} \tag{12.8}$$

with the convention that $\nu_{\Lambda_0} \equiv \mathbf{I}$ is the identity operator and $\Lambda_{l+1} = \Lambda_l \cup \{j_{l+1}\}$. Using this together with the previous inequality, we get the following assertion.

**Lemma 12.4.**

$$E_{\Lambda_{k(i)}}\left|F - E_{\Lambda_{k(i)}}F\right|^2 \leqslant e^{4\phi|\Lambda_{k(i)}\cap\Lambda_F|}|\Lambda_{k(i)}\cap\Lambda_F| \sum_{j\in\Lambda_{k(i)}} E_{\Lambda_{k(i)}}\left(\nu_j|F-\nu_jF|^p\right)^{\frac{2}{p}}.$$

Combining with (12.7), we arrive at

$$\begin{aligned}2\left(\nu_i\left|[E_{\widetilde{\Lambda}^{(i)}},\nu_i]F\right|^p\right)^{\frac{2}{p}} \leqslant 2\sum_{k(i)}&\left[\operatorname{osc}_{\Lambda_{k(i)}\cap\Lambda_F}\left(E_{\Lambda_{k(i)}\setminus\Lambda_F}(D_i)\right)\right]^2 \\ &\cdot e^{4\phi[|\Lambda_{k(i)}\cap\Lambda_F|+\frac{1}{2}}|\Lambda_{k(i)}\cap\Lambda_F| \\ &\cdot\sum_{j\in\Lambda} E_iE_{\Lambda_{k(i)}}\left(\nu_j|F-\nu_jF|^p\right)^{\frac{2}{p}}. \qquad (12.9)\end{aligned}$$

This ends the estimates for the second term on the right-hand side of (12.1).

Using (12.2) and (12.9), we find

$$\begin{aligned}&\left(\nu_i\left|E_{\widetilde{\Gamma}_n^{(i)}}E_{\widetilde{\Lambda}^{(i)}}F-\nu_iE_{\widetilde{\Gamma}_n^{(i)}}E_{\widetilde{\Lambda}^{(i)}}F\right|^p\right)^{\frac{2}{p}} \leqslant 2e^{6||u_i||}E_{\widetilde{\Lambda}^{(i)}}\left(\nu_i\left|F-\nu_iF\right|^p\right)^{\frac{2}{p}} \\ &\quad+2\sum_{k(i)}\left[\operatorname{osc}_{\Lambda_{k(i)}\cap\Lambda_F}\left(E_{\Lambda_{k(i)}\setminus\Lambda_F}(D_i)\right)\right]^2\cdot e^{4\phi[|\Lambda_{k(i)}\cap\Lambda_F|+\frac{1}{2}]}|\Lambda_{k(i)}\cap\Lambda_F| \\ &\quad\cdot\sum_{j\in\Lambda_{k(i)}}E_{\widetilde{\Gamma}_n^{(i)}}E_iE_{\Lambda_{k(i)}}\left(\nu_j|F-\nu_jF|^p\right)^{\frac{2}{p}}.\end{aligned}$$

From this we conclude that

$$\begin{aligned}\sum_{i\in\Gamma_{n+1}}\mu\left(\nu_i\left|E_{\Gamma_n}F-\nu_iE_{\Gamma_n}F\right|^p\right)^{\frac{2}{p}} &\leqslant \sum_{i\in\Gamma_{n+1}}2e^{6||u_i||}\mu\left(\nu_i\left|F-\nu_iF\right|^p\right)^{\frac{2}{p}} \\ &+2\sum_{i\in\Gamma_{n+1}}\sum_{k(i)}\left[\operatorname{osc}_{\Lambda_{k(i)}\cap\Lambda_F}\left(E_{\Lambda_{k(i)}\setminus\Lambda_F}(D_i)\right)\right]^2 \\ &\cdot e^{4\phi[|\Lambda_{k(i)}\cap\Lambda_F|+\frac{1}{2}}|\Lambda_{k(i)}\cap\Lambda_F|\cdot\sum_{j\in\Lambda_{k(i)}}\mu\left(\nu_j|F-\nu_jF|^p\right)^{\frac{2}{p}}.\end{aligned}$$

Lemma 12.3 is proved. □

Applying iteratively Lemma 12.3, we arrive at the following result.

**Proposition 12.5.** *Suppose that $\Gamma_n$, $n\in\mathbb{N}$, is a periodic sequence of period $N$ such that $\bigcup_{l=1,\ldots,N}\Gamma_l=\mathcal{R}$. Then there exists a constant $C\in(0,\infty)$ such that for any $p\in(2,\infty)$ and any $1\leqslant n\leqslant N-1$*

$$\sum_{i\in\Gamma_{n+1}} \mu\left(\nu_i|\boldsymbol{\Pi}_n f - \nu_i \boldsymbol{\Pi}_n f|^p\right)^{\frac{2}{p}}$$
$$\leqslant \sum_{i\in\Gamma_{n+1}} \sum_{j_n\in\Gamma_n\setminus\Gamma_{n+1},\dots,j_1\in\Gamma_1\setminus\Gamma_2} \eta_{ij_n}\eta_{j_nj_{n-1}}\dots\eta_{j_2j_1}\mu\left(\nu_j|f-\nu_j f|^p\right)^{\frac{2}{p}}$$
$$\leqslant C\sum_{i\in\mathcal{R}} \mu\left(\nu_j|f-\nu_j f|^p\right)^{\frac{2}{p}} .$$

*Moreover, for $n = N$*

$$\sum_{i\in\Gamma_{n+1}} \mu\left(\nu_i|\boldsymbol{\Pi}_N F - \nu_i \boldsymbol{\Pi}_N F|^p\right)^{\frac{2}{p}} \leqslant \lambda \sum_{j\in\mathcal{R}} \mu\left(\nu_j|F-\nu_j F|^p\right)^{\frac{2}{p}}$$

*with*

$$\lambda \equiv \sup_{j\in\mathcal{R}} \sum_{i\in\Gamma_{n+1}} \sum_{j_n\in\Gamma_n\setminus\Gamma_{n+1},\dots,j_1\in\Gamma_1\setminus\Gamma_2} \eta_{ij_n}\eta_{j_nj_{n-1}}\dots\eta_{j_2j}. \tag{12.10}$$

*Therefore, for any $n\in\mathbb{N}$*

$$\sum_{i\in\mathcal{R}} \mu\left(\nu_i|\boldsymbol{\Pi}_n f - \nu_i \boldsymbol{\Pi}_n f|^p\right)^{\frac{2}{p}} \leqslant C\lambda^{[\frac{n}{N}]} \sum_{i\in\mathcal{R}} \mu\left(\nu_j|f-\nu_j f|^p\right)^{\frac{2}{p}} ,$$

*where $[n/N]$ is the integer part of $n/N$, with some constant $C\in(0,\infty)$.*

Remark 12.2. Because of our assumption that conditional expectations satisfy local Markov property, $\lambda$ is defined by a finite sum and therefore is finite.

**Lemma 12.6.** *There exists a constant $\gamma_0 \in (0,\infty)$ such that for any $p\in(2,\infty)$ and $s\in(0,s_0)$*

$$\mu\left|\nabla_i E_{\Gamma_n} F\right|^2 \leqslant \gamma_0\mu\left|\nabla_i F\right|^2 + \widetilde{\beta}(s)\sum_j \boldsymbol{\eta}_{ij}\mu|\nabla_j F|^2 + s\sum_j \boldsymbol{\eta}_{ij}\mu\left(\nu_j|F-\nu_j F|^p\right)^{\frac{2}{p}}$$

*with*

$$\boldsymbol{\eta}_{ij} \equiv \sum_{k(i):j\in\Lambda_{k(i)}} \gamma_0 \left\| \operatorname{osc}_{\Lambda_{k(i)}\cap\Lambda_F}\left(E_{\Lambda_{k(i)}\setminus\Lambda_F}(\nabla_i U_{\Lambda_{k(i)}})\right)\right\|_\infty^2 \cdot e^{4\phi|\Lambda_{k(i)}\cap\Lambda_F|}|\Lambda_{k(i)}\cap\Lambda_F|$$

*defined for $j\in\widetilde{\Lambda}^{(i)}$ and zero otherwise.*

*Proof.* For $i\in\Gamma_{n+1}$ let $\Lambda_{k(i)}\subset\Gamma_n$ be such that

$$i\in\partial_R\Lambda_{k(i)} \equiv \{j\in\Lambda^c_{k(i)} :\ \operatorname{dist}(j,\Lambda_{k(i)})\leqslant R\}.$$

Let $\widetilde{\Gamma}_n^{(i)} \equiv \Gamma_n \setminus \widetilde{\Lambda}^{(i)}$, where $\widetilde{\Lambda}^{(i)} \equiv \cup \Lambda_{k(i)}$. Note that

$$E_{\Gamma_n} F = E_{\widetilde{\Gamma}_n^{(i)}} E_{\widetilde{\Lambda}^{(i)}} F, \quad \nabla_i E_{\widetilde{\Gamma}_n^{(i)}} = E_{\widetilde{\Gamma}_n^{(i)}} \nabla_i.$$

Hence

$$\nabla_i E_{\Gamma_n} F = \nabla_i E_{\widetilde{\Gamma}_n^{(i)}} E_{\widetilde{\Lambda}^{(i)}} F = E_{\widetilde{\Gamma}_n^{(i)}} \nabla_i E_{\widetilde{\Lambda}^{(i)}} F.$$

On the other hand, we have

$$\nabla_i E_{\widetilde{\Lambda}^{(i)}} F = E_{\widetilde{\Lambda}^{(i)}} \nabla_i F + [\nabla_i, E_{\widetilde{\Lambda}^{(i)}}] F,$$

where $[\nabla_i, E_{\widetilde{\Lambda}^{(i)}}] F \equiv \nabla_i E_{\widetilde{\Lambda}^{(i)}} F - E_{\widetilde{\Lambda}^{(i)}} \nabla_i F$. We note that

$$[\nabla_i, E_{\widetilde{\Lambda}^{(i)}}] F = \sum_{k(i)} E_{\widetilde{\Lambda}^{(i)}} \big( E_{\Lambda_{k(i)}} \big( F; \nabla_i U_{\Lambda_{k(i)}} \big) \big),$$

where

$$E_{\Lambda_{k(i)}} \big( F; \nabla_i U_{\Lambda_{k(i)}} \big) \equiv E_{\Lambda_{k(i)}} \big( F \cdot \nabla_i U_{\Lambda_{k(i)}} \big) - E_{\Lambda_{k(i)}} (F) E_{\Lambda_{k(i)}} \big( \nabla_i U_{\Lambda_{k(i)}} \big).$$

If $F$ depends on variables in $\Lambda_{k(i)} \cap \Lambda_F$, we have

$$\begin{aligned}
&\big| E_{\Lambda_{k(i)}} \big( F; \nabla_i U_{\Lambda_{k(i)}} \big) \big| = \big| E_{\Lambda_{k(i)}} \big( (F - E_{\Lambda_{k(i)}} F) \cdot E_{\Lambda_{k(i)} \setminus \Lambda_F} (\nabla_i U_{\Lambda_{k(i)}}) \big) \big| \\
&\quad \leqslant \operatorname{osc}_{\Lambda_{k(i)} \cap \Lambda_F} \big( E_{\Lambda_{k(i)} \setminus \Lambda_F} (\nabla_i U_{\Lambda_{k(i)}}) \big) \cdot E_{\Lambda_{k(i)}} \big| F - E_{\Lambda_{k(i)}} F \big|.
\end{aligned}$$

Thus, in this case,

$$\begin{aligned}
\big| \nabla_i E_{\Gamma_n} F \big| &\leqslant E_{\Gamma_n} \big| \nabla_i F \big| + E_{\widetilde{\Gamma}_n^{(i)}} \big| [\nabla_i, E_{\widetilde{\Lambda}^{(i)}}] F \big| \\
&\leqslant E_{\Gamma_n} \big| \nabla_i F \big| + \sum_{k(i)} E_{\Gamma_n} \big| \big( E_{\Lambda_{k(i)}} \big( F; \nabla_i U_{\Lambda_{k(i)}} \big) \big) \big| \\
&\leqslant E_{\Gamma_n} \big| \nabla_i F \big| + \sum_{k(i)} E_{\Gamma_n} \operatorname{osc}_{\Lambda_{k(i)} \cap \Lambda_F} \big( E_{\Lambda_{k(i)} \setminus \Lambda_F} (\nabla_i U_{\Lambda_{k(i)}}) \big) \\
&\qquad \cdot E_{\Lambda_{k(i)}} \big| F - E_{\Lambda_{k(i)}} F \big|.
\end{aligned}$$

Therefore, there exists a constant $\gamma_0 \in (0, \infty)$ depending only on the number of $k(i)$'s such that

$$\begin{aligned}
\mu \big| \nabla_i E_{\Gamma_n} F \big|^2 \leqslant \gamma_0 \mu \big| \nabla_i F \big|^2 + \sum_{k(i)} \gamma_0 \big\| \operatorname{osc}_{\Lambda_{k(i)} \cap \Lambda_F} \big( E_{\Lambda_{k(i)} \setminus \Lambda_F} (\nabla_i U_{\Lambda_{k(i)}}) \big) \big\|_\infty^2 \\
\cdot \mu \big( E_{\Lambda_{k(i)}} \big| F - E_{\Lambda_{k(i)}} F \big|^2 \big).
\end{aligned}$$

Using Lemma 12.1, we obtain

$$\mu\left|\nabla_i E_{\Gamma_n} F\right|^2 \leqslant \gamma_0 \mu\left|\nabla_i F\right|^2 + \widetilde{\beta}(s) \sum_j \boldsymbol{\eta}_{ij} \mu |\nabla_j F|^2 \\ + s \sum_j \boldsymbol{\eta}_{ij} \mu\left(\nu_j |F - \nu_j F|^p\right)^{\frac{2}{p}},$$

where $\boldsymbol{\eta}_{ij}$, for $i \in \mathbb{Z}^d \setminus \widetilde{\Lambda}^{(i)}$, dist $(i, \widetilde{\Lambda}^{(i)}) \leqslant R$, $j \in \widetilde{\Lambda}^{(i)}$, are defined by

$$\boldsymbol{\eta}_{ij} \equiv \sum_{k(i): j \in \Lambda_{k(i)}} \gamma_0 \left\| \operatorname{osc}_{\Lambda_{k(i)} \cap \Lambda_F} \left(E_{\Lambda_{k(i)} \setminus \Lambda_F} (\nabla_i U_{\Lambda_{k(i)}})\right)\right\|_\infty^2 \\ \cdot e^{4\phi |\Lambda_{k(i)} \cap \Lambda_F|} |\Lambda_{k(i)} \cap \Lambda_F|.$$

The lemma is proved. □

**Proposition 12.7.** *Suppose that $\Gamma_n$, $n \in \mathbb{N}$, is a periodic sequence of period $N$ such that $\bigcup_{l=1\dots N} \Gamma_l = \mathcal{R}$ and so for any $i \in \mathcal{R}$ there exists $1 \leqslant l(i) \leqslant N$ for which $\nabla_i E_{\Gamma_{l(i)}} f = 0$. Then for any $p \in (2, \infty)$ and any $1 \leqslant n \leqslant N - 1$*

$$\mu \left|\nabla_i E_{\Gamma_n} \dots E_{\Gamma_1} F\right|^2 \leqslant X(s) \mu \left|\nabla_i F\right|^2 + s Z(s) \sum_j \mu \left(\nu_j |F - \nu_j F|^p\right)^{\frac{2}{p}}$$

*with $X(s) \equiv X(1+\widetilde{\beta}(s)^{N-1})$ and $Z(s) \equiv Z(1+\widetilde{\beta}(s)^{N-1})$ with some constants $X, Z > 0$. Moreover, for $n \geqslant N$*

$$\mu \left|\nabla_i \boldsymbol{\Pi}_n f\right|^2 \leqslant s Z(s) \lambda^{\frac{n}{N}} \sum_{j \in \mathcal{R}} \mu \left(\nu_j |f - \nu_j f|^p\right)^{\frac{2}{p}}.$$

*Proof.* For $n \leqslant N$, by Lemma 12.6, we have

$$\mu \left|\nabla_{\Gamma_{n+1}} \boldsymbol{\Pi}_n F\right|^2 = \sum_{i \in \Gamma_{n+1}} \mu \left|\nabla_i \boldsymbol{\Pi}_n F\right|^2 \\ \leqslant \sum_{i, j_n} \mathbf{1}_{\{i \in \Gamma_{n+1} \setminus \Gamma_n\}} \boldsymbol{A}^{(n+1,n)}(s) \mathbf{1}_{\{j_n \in \Gamma_n \setminus \Gamma_{n-1}\}} \mu \left|\nabla_{j_n} \boldsymbol{\Pi}_{n-1} F\right|^2 \\ + s \sum_{i, j_n} \mathbf{1}_{\{i \in \Gamma_{n+1} \setminus \Gamma_n\}} \boldsymbol{\eta}^{(n+1,n)} \mathbf{1}_{\{j_n \in \Gamma_n \setminus \Gamma_{n-1}\}} \mu \left(\nu_{j_n} \left|\boldsymbol{\Pi}_{n-1} F - \nu_{j_n} \boldsymbol{\Pi}_{n-1} F\right|^p\right)^{\frac{2}{p}},$$

where

$$\boldsymbol{A}^{(n+1,n)}(s)_{ij} \equiv \left(a\boldsymbol{I} + \boldsymbol{\eta}^{(n+1,n)}\right)_{ij} \equiv \left(\gamma_0 \delta_{ij} + \widetilde{\beta}(s) \eta_{ij}^{(n+1,n)}\right),$$

where $\mathbf{1}_{\{j_n \in \Gamma_n \setminus \Gamma_{n-1}\}}$ denotes the characteristic function of the set $\Gamma_n \setminus \Gamma_{n-1}$ and $\eta_{ij}^{(n+1,n)}$ is provided by Lemma 12.6 (with $i \in \Gamma_{n+1} \setminus \Gamma_n$ and $j \in \Gamma_n$). By induction, we arrive at the bound

$$\mu\left|\nabla_{\Gamma_{n+1}}\boldsymbol{\Pi}_n F\right|^2 \leqslant \sum_{i,j}\boldsymbol{\Theta}_{ij}^{(n,1)}\mu\left|\nabla_j F\right|^2 + s\sum_{k=1\ldots n}\sum_{i,j}\boldsymbol{\Upsilon}_{ij}^{(k)}\mu\left(\nu_j\left|\boldsymbol{\Pi}_{n-k}F-\nu_j\boldsymbol{\Pi}_{n-k}F\right|^p\right)^{\frac{2}{p}},$$

where

$$\boldsymbol{\Theta}^{(n,m)} \equiv \mathbf{1}_{\{i\in\Gamma_{n+1}\setminus\Gamma_n\}}\boldsymbol{A}^{(n+1,n)}(s)\mathbf{1}_{\{j_n\in\Gamma_n\setminus\Gamma_{n-1}\}}\boldsymbol{A}^{(n,n-1)}(s)\mathbf{1}_{\{j_{n-1}\in\Gamma_{n-1}\setminus\Gamma_{n-2}\}} \ldots\mathbf{1}_{\{j_{m+1}\in\Gamma_{m+1}\setminus\Gamma_m\}}\boldsymbol{A}^{(m+1,m)}(s)\mathbf{1}_{\{j_m\in\Gamma_m\setminus\Gamma_{m-1}\}}$$

with the convention that $\Gamma_0 \equiv \varnothing$, and

$$\boldsymbol{\Upsilon}^{(k)} \equiv \boldsymbol{\Theta}^{(n,n-k)}\cdot\boldsymbol{\eta}^{(n+1-k,n-k)}\mathbf{1}_{\{j\in\Gamma_{n-k}\setminus\Gamma_{n-k-1}\}}$$

with the convention that $\boldsymbol{\Theta}^{(n,n)} \equiv \mathbf{1}_{\{\Gamma_{n+1}\setminus\Gamma_n\}}$.

Using Proposition 12.5, we can simplify the above estimate as follows:

$$\mu\left|\nabla_{\Gamma_{n+1}}\boldsymbol{\Pi}_n F\right|^2 \leqslant \sum_{i,j}\boldsymbol{\Theta}_{ij}^{(n,1)}\mu\left|\nabla_j F\right|^2 + s\sum_{i,j}\boldsymbol{\Upsilon}_{ij}^{(n)}\mu\left(\nu_j\left|F-\nu_j F\right|^p\right)^{\frac{2}{p}},$$

where

$$\boldsymbol{\Upsilon}^{(n)} \equiv \sum_{k=1\ldots n}\boldsymbol{\Upsilon}^{(k)}\mathbf{1}_{\{j\in\Gamma_{n-k}\setminus\Gamma_{n-k-1}\}}\boldsymbol{\eta}^{(n-k,n-k-1)}\ldots\mathbf{1}_{\{j\in\Gamma_2\setminus\Gamma_1\}}\boldsymbol{\eta}^{(2,1)}.$$

We note that there is a constant $X\in(0,\infty)$ such that for $n\leqslant N-1$

$$\sup_j\sum_i\boldsymbol{\Theta}_{ij}^{(n,1)} \leqslant X(1+\widetilde{\beta}(s)^{N-1}).$$

If we assume that for each $i$ there is an $l\leqslant N$ such that $\nabla_i E_{\Gamma_l} = \mathbf{0}$, then we get

$$\mu\left|\nabla_{\Gamma_{N+1}}\boldsymbol{\Pi}_N F\right|^2 \leqslant s\sum_{i,j}\boldsymbol{\Upsilon}_{ij}^{(N)}\mu\left(\nu_j\left|F-\nu_j F\right|^p\right)^{\frac{2}{p}}$$

Since

$$\sup_j\sum_i\boldsymbol{\Upsilon}_{ij}^{(n)} \leqslant C'(1+\widetilde{\beta}(s)^{N-1})$$

with some constant $C'\in(0,\infty)$ independent of $n\leqslant N$, we get

$$\mu\left|\nabla_{\Gamma_{N+1}}\boldsymbol{\Pi}_N F\right|^2 \leqslant s\;C'(1+\widetilde{\beta}(s)^{N-1})\sum_j\mu\left(\nu_j\left|F-\nu_j F\right|^p\right)^{\frac{2}{p}}.$$

As a consequence for $n\geqslant N$, setting $F\equiv\boldsymbol{\Pi}_{n-N}f$ and using Proposition 12.5, we conclude that

$$\mu \left|\nabla_{\Gamma_{n+1}} \boldsymbol{\Pi}_n f\right|^2 \leqslant s\ Z(1+\widetilde{\beta}(s)^{N-1})\lambda^{[\frac{n}{N}]} \sum_{j\in\mathcal{R}} \mu\left(\nu_j\left|f-\nu_j f\right|^p\right)^{\frac{2}{p}},$$

where $Z \equiv CC'$. □

**Theorem 12.8.** *Suppose that $\Gamma_n$, $n \in \mathbb{N}$, is a periodic sequence of period $N$ such that $\bigcup_{l=1\ldots N} \Gamma_l = \mathcal{R}$ and so for any $i \in \mathcal{R}$ there exists $1 \leqslant l(i) \leqslant N$ for which $\nabla_i E_{\Gamma_{l(i)}} f = 0$. Suppose that the parameter $\lambda$ introduced in Proposition 12.5 satisfies $\lambda \in (0,1)$. Then for any $p \in (2,\infty)$*

$$\mu\left|f-\mu f\right|^2 \leqslant \beta(s)\mu\left|\nabla f\right|^2 + s\sum_{i\in\mathcal{R}} \mu\left(\nu_i|f-\nu_i f|^p\right)^{\frac{2}{p}}$$

*with*

$$\beta(s) \equiv \widetilde{\beta}(\vartheta^{-1}(s))X(\vartheta^{-1}(s))N \equiv X(\widetilde{\beta}(\vartheta^{-1}(s)) + \widetilde{\beta}(\vartheta^{-1}(s))^N)N$$

*for $s \in (0, \vartheta(s_0))$, where $\vartheta(s) \equiv sN(C+\widetilde{\beta}(s)Z(s))(1-\lambda)^{-1}$ with $Z(s) \equiv Z(1+\widetilde{\beta}(s)^{N-1})$, with some constants $X, Z, C > 0$.*

*Proof.* By Lemma 12.2, we have

$$\begin{aligned}\mu E_{\Gamma_{n+1}}\left|\boldsymbol{\Pi}_n f - E_{\Gamma_{n+1}}\boldsymbol{\Pi}_n f\right|^2 &\leqslant \widetilde{\beta}(s)\mu\left|\nabla_{\Gamma_{n+1}}\boldsymbol{\Pi}_n f\right|^2 \\ &\quad + s\sum_{i\in\Gamma_{n+1}} \mu\left(\nu_i|\boldsymbol{\Pi}_n f - \nu_i \boldsymbol{\Pi}_n f|^p\right)^{\frac{2}{p}}.\end{aligned}$$

Hence, by Propositions 12.5, 12.7 and Lemma 12.6, for $n \leqslant N-1$ we have

$$\begin{aligned}\mu E_{\Gamma_{n+1}}\left|\boldsymbol{\Pi}_n f - E_{\Gamma_{n+1}}\boldsymbol{\Pi}_n f\right|^2 &\leqslant \widetilde{\beta}(s)X(s)\mu\left|\nabla f\right|^2 \\ &\quad + s(C+\widetilde{\beta}(s)Z(s))\sum_{i\in\mathcal{R}} \mu\left(\nu_i|f-\nu_i f|^p\right)^{\frac{2}{p}},\end{aligned}$$

while for $n \geqslant N$ we have

$$\mu E_{\Gamma_{n+1}}\left|\boldsymbol{\Pi}_n f - E_{\Gamma_{n+1}}\boldsymbol{\Pi}_n f\right|^2 \leqslant s(C+\widetilde{\beta}(s)Z(s))\lambda^{[\frac{n}{N}]}\sum_{i\in\mathcal{R}} \mu\left(\nu_i|f-\nu_i f|^p\right)^{\frac{2}{p}}.$$

Thus, provided that $\lambda \in (0,1)$, we arrive at

$$\begin{aligned}\mu\left|f-\mu f\right|^2 &= \sum_{n\in\mathbb{Z}^+} \mu E_{\Gamma_{n+1}}\left|\boldsymbol{\Pi}_n f - E_{\Gamma_{n+1}}\boldsymbol{\Pi}_n f\right|^2 \\ &\leqslant \widetilde{\beta}(s)X(s)N\mu\left|\nabla f\right|^2 + sN(C+\widetilde{\beta}(s)Z(s))(1-\lambda)^{-1}\sum_{i\in\mathcal{R}} \mu\left(\nu_i|f-\nu_i f|^p\right)^{\frac{2}{p}}.\end{aligned}$$

The theorem is proved. □

**Examples.** Suppose that $\mathcal{R} = \mathbb{Z}^d$, $d \in \mathbb{N}$. Then the corresponding covering $\Gamma_n$, $n = 1 \dots 2^d$, was introduced as a collection of suitable translates a sufficiently large cube $\Lambda_0$ for $d = 1$ in [40] and for general $d$ in [36]. In the case where the local specification $E_\Lambda$, $\Lambda \subset\subset \mathbb{Z}^d$ satisfies the *strong mixing condition* (for cubes)

$$|E_\Lambda(f; g)| \leqslant \text{Const } |||f||| \cdot |||g|||e^{-M \ \text{dist}\,(\Lambda_f, \Lambda_g)}$$

with some constant $M \in (0, \infty)$ independent of size of the cube, one shows (cf., for example, [40, 36, 21]) that, starting with a sufficiently large cube $\Lambda_0$, one can achieve $\lambda \in (0, 1)$. In our case, the strong mixing condition holds at least for finite range sufficiently small interactions $u_\Lambda$.

In our setup, by Corollary 8.6 and Lemma 12.1, $\overline{\beta}(s) \equiv C_0(\log(1/s))^\delta$ with some positive $C_0$ and $\delta \in (0, \infty)$ for all sufficiently small $s > 0$. Hence

$$\beta(s) = C(\log(1/s))^{N\delta}$$

with some positive constant $C$ for all sufficiently small $s > 0$. Thus, the above considerations (cf. Theorem 12.8) apply and we have the following result.

**Theorem 12.9.** *Let $\mu$ be a Gibbs measure on $\mathbb{R}^{\mathbb{Z}^d}$ corresponding to the reference product measure $\mu_0 \equiv \nu_0^{\otimes \mathbb{Z}^d}$, where the probability measure $d\nu_0 \equiv \frac{1}{Z}\exp\{-V\}dx$ on real line is defined with $V \equiv \varsigma(1 + x^2)^{\frac{\alpha}{2}}$, with $0 < \alpha \leqslant 1$, $\varsigma \in (0, \infty)$, and a local finite range smooth interaction $u_\Lambda$, $\Lambda \subset\subset \mathbb{Z}^d$, which is sufficiently small or more generally such that the Strong Mixing Condition holds.*

*Then $\mu$ satisfies the weak Poincaré inequality*

$$\mu\,|f - \mu f|^2 \leqslant \beta(s)\mu|\nabla f|^2 + sA_p(f)^2$$

*with $\beta(s) \equiv C(\log(1/s))^{N\delta}$ with some positive constant $C$ and $N = 2^d$ for all $s \in (0, \overline{s})$ for some $\overline{s} > 0$. Hence there exists $\varepsilon \in (0, 1)$ and constants $c, H \in (0, \infty)$ such that the semigroup $P_t \equiv e^{t\nabla^*\nabla}$ (with the generator corresponding to the Dirichlet form $\mu|\nabla f|^2$) satisfies*

$$\mu(P_t f - \mu f)^2 \leqslant e^{-ct^\varepsilon} H \left(\mu\,(f - \mu f)^2 + (\mu\,|f - \mu f|^p)^{\frac{2}{p}} + |||f|||^2\right)$$

*for each cylinder function for which the right hand side is well defined (with a constant $H \in (0, \infty)$ dependent on $\Lambda_f$).*

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