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PERTURBATIONS IN THE GAUSSIAN ISOPERIMETRIC INEQUALITY

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An isoperimetric inequality of Gaussian type is derived for the class of probability measures on the Euclidean space, having perturbed log-concave densities with respect to the standard Gaussian measure. Bibliography: 23 *titles.*

1. Introduction

Let γ_n denote the standard Gaussian measure on the Euclidean space $\mathbf{R}^n$ with density

$$\frac{d\gamma_n(x)}{dx} = \frac{1}{(2\pi)^{n/2}}\, e^{-|x|^2/2}, \quad x \in \mathbf{R}^n.$$

The Gaussian isoperimetric inequality states that, for any measurable set $A \subset \mathbf{R}^n$ and any $h > 0$,

$$\gamma_n(A^h) \geqslant \Phi\left(\Phi^{-1}(\gamma_n(A)) + h\right), \tag{1.1}$$

where

$$A^h = \{x \in \mathbf{R}^n : \exists y \in A,\ |x-y| < h\}$$

denotes an open h-neighborhood of A (for the Euclidean distance). Hereinafter, we use the standard notation

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \quad \Phi(x) = \int\limits_{-\infty}^{x} \varphi(y)\, dy \quad (-\infty \leqslant x \leqslant +\infty),$$

for the marginal density and the marginal distribution function of γ_n with the inverse function $\Phi^{-1} : [0,1] \to [-\infty, +\infty]$.

In other words, among all subsets A of $\mathbf{R}^n$ with a fixed measure $t = \gamma_n(A)$, the value $\gamma_n(A^h)$ attains minimum for half-spaces of measure t.

Letting $h \to 0$ in (1.1), in the limit one arrives at an equivalent isoperimetric inequality, which may be written as

$$\gamma_n^+(A) \geqslant I(\mu(A)), \tag{1.2}$$

where

$$\gamma_n^+(A) = \liminf_{h\to 0} \frac{\gamma_n(A^h) - \gamma_n(A)}{h}$$

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is the Gaussian perimeter of A, and where

$$I(t) = \varphi(\Phi^{-1}(t)), \quad 0 \leqslant t \leqslant 1,$$

is the isoperimetric profile (also called the isoperimetric function or the area minimizing function) for the measure γ_n.

The inequality (1.1) was discovered in the mid 1970's independently by Sudakov and Cirel'son [**1**], and Borell [**2**]. It has become of a fundamental importance in the theory of Gaussian random processes, and it is not surprising that for many years this result continued to attract a lot of attention. Nowadays several different proofs of (1.1) are known; let us mentioned them.

1. The original proof of [**1**] and [**2**] based on the isoperimetric property of balls on the sphere (a theorem due to P. Lévy and E. Schmidt).
2. The proof based on the Brunn–Minkowski type inequality due to Ehrhard (cf. [**3**]–[**7**]).
3. The semigroup proof involving Ornstein–Uhlenbeck operators [**8, 9**].
4. The proof based on a certain functional form of the isoperimetric inequality on the discrete cube [**10**].
5. The proof based on the localization lemma of Lovász–Simonovits [**11**].

Some of the developed approaches allowed one to involve in (1.1)-(1.2) different non-Gaussian probability measures. In particular, as was established by Bakry and Ledoux [**8**], one has a similar isoperimetric inequality of Gaussian type

$$\mu(A^h) \geqslant \Phi\left(\Phi^{-1}(\mu(A)) + h\right), \tag{1.3}$$

for any probability measure μ on $\mathbf{R}^n$, which has a log-concave density with respect γ_n. Equivalently, it is the case where μ has density of the form

$$\frac{d\mu(x)}{dx} = e^{-\frac{1}{2}|x|^2 - v(x)}, \quad x \in \Omega, \tag{1.4}$$

with some convex function $v : \Omega \to \mathbf{R}$, defined on an open convex set Ω in $\mathbf{R}^n$ (bounded or not). A different proof of this result is given in [**11**]. On the other hand, Caffarelli [**12**] showed that any such measure μ represents a contraction (i.e., the image under a Lipschitz map with Lipschitz constant at most 1) of the measure γ_n. Hence the inequality (1.3) for μ, having a log-concave density with respect γ_n, may also be derived from the purely Gaussian case (1.1).

2. Perturbations

The goal of this paper is to extend the isoperimetric inequality (1.3) to more general probability measures, which have perturbed log-concave densities with respect to the standard Gaussian measure.

Theorem 2.1. *Let μ be a probability measure on an open convex set Ω in $\mathbf{R}^n$ with density* (1.4)*, where v is a continuous function on Ω such that*

$$v^*(x) \leqslant v(x) \leqslant v^*(x) + c, \quad x \in \Omega, \tag{2.1}$$

for some convex function v^* on Ω and some constant $c \geqslant 0$. Then, for any measurable set $A \subset \mathbf{R}^n$ and $h > 0$,

$$\mu(A^h) \geqslant \Phi\left(\Phi^{-1}(\mu(A)) + e^{-c}h\right). \tag{2.2}$$

The statement may be sharpened by considering various functional forms for (2.2). In particular, we have the following assertion.

Theorem 2.2. *Under the assumption of Theorem* 1.1, *for any smooth function* f *on* $\mathbf{R}^n$ *with values in* $[0,1]$,

$$I(\mathbf{E}f) \leqslant \mathbf{E}\sqrt{I(f)^2 + e^{2c}\,|\nabla f|^2}. \tag{2.3}$$

Here $|\nabla f|$ denotes the Euclidean length of the gradient of f, and the expectation

$$\mathbf{E}f = \int f\,d\mu$$

is understood with respect to the measure μ. By a simple approximation argument, the inequality (2.3) may be extended to the class of all locally Lipschitz functions $f : \mathbf{R}^n \to [0,1]$ with the generalized modulus of the gradient, defined by

$$|\nabla f(x)| = \limsup_{y\to x} \frac{|f(x)-f(y)|}{|x-y|}, \quad x \in \mathbf{R}^n. \tag{2.4}$$

Such functions are differentiable almost everywhere, and for all points x of differentiability of f, (2.4) leads to the usual definition

$$|\nabla f(x)|^2 = \sum_{i=1}^n \Big|\frac{\partial f(x)}{\partial x_i}\Big|^2.$$

In view of the elementary bound

$$\sqrt{a^2+b^2} \leqslant |a| + |b| \quad (a, b \in \mathbf{R}),$$

(2.3) yields

$$I(\mathbf{E}f) - \mathbf{E}I(f) \leqslant e^c\,\mathbf{E}\,|\nabla f| \tag{2.5}$$

in the same class of functions. Moreover, approximating by smooth functions f the indicator functions 1_A of Borel subsets of the Euclidean space, (2.5) turns into the isoperimetric inequality for μ-perimeter (like (1.2)),

$$\mu^+(A) \geqslant e^{-c}\,I(\mu(A)). \tag{2.6}$$

The latter may easily be "integrated" with respect to the parameter h to obtain (2.2) (cf., for example, [**13, 14**] for details). As for the converse implication, it is also simple, so the inequalities (2.2), (2.5), and (2.6) are equivalent.

However, the functional inequality (2.3) is much more delicate, and we are not sure that it can be obtained on the basis of (2.2) when dealing with general measures.

The main advantage of the functional form (2.3) over (2.2) is its tensorization property. Namely (cf. [**10**]), starting with a probability measure μ on $\mathbf{R}^n$ satisfying (2.3) for a given continuous function $I(t) \geqslant 0$, we obtain automatically a similar inequality (i.e., (2.3) and therefore (2.2) as a consequence) for all product measures

$$\mu^N = \mu \otimes \cdots \otimes \mu$$

on $\mathbf{R}^{nN}$ with the same function I and the same constant e^c.

Another advantage of (2.3) is that it contains a number of canonical analytic inequalities. For example, applying it to functions of the form εf with a bounded smooth $f \geqslant \varepsilon_0 > 0$ and $\varepsilon \to 0$, and using the asymptotic

$$I(t) \sim t\sqrt{2\log(1/t)}$$

for small values of t, we get in the limit

$$\mathbf{E} f \log f - \mathbf{E} f \, \log \mathbf{E} f \leqslant \frac{e^{2c}}{2} \, \mathbf{E} \, \frac{|\nabla f|^2}{f}.$$

By a simple approximation, this inequality extends to arbitrary smooth functions $f > 0$ on Ω with finite $\mathbf{E} f$. Replacing f with f^2, we obtain a logarithmic Sobolev inequality in the standard form

$$\mathbf{E} f^2 \log f^2 - \mathbf{E} f^2 \, \log \mathbf{E} f^2 \leqslant 2e^{2c} \, \mathbf{E} \, |\nabla f|^2, \tag{2.7}$$

which holds in the class of all smooth f on Ω with finite second moment $\mathbf{E} f^2$. (In fact, if the right-hand side of (2.7) is finite, $\mathbf{E} f^2$ has to be finite, as well.)

Applying (2.7) to $f + C$ with $C \to +\infty$, we obtain a Poincaré-type inequality

$$\mathbf{E} f^2 - (\mathbf{E} f)^2 \leqslant e^{2c} \, \mathbf{E} \, |\nabla f|^2. \tag{2.8}$$

It may also be obtained directly from the functional form (2.3) by applying it to functions of the form $t + \varepsilon f$ with an arbitrary fixed $t \in (0,1)$ and $\varepsilon \to 0$ (in this derivation, one should use the fact that the Gaussian isoperimetric function satisfies the differential equation $I''(t) = -1/I(t)$ in $0 < t < 1$).

In the Gaussian case $\mu = \gamma_n$, we have $c = 0$, and then both (2.7) and (2.8) are well known (the logarithmic Sobolev inequality (2.7) is due to Gross [**15**]). More generally, we have $c = 0$, when μ is log-concave with respect to γ_n, i.e., when it has density

$$p(x) = e^{-\frac{1}{2}\,|x|^2 - v(x)}$$

with convex v [**8**]. In particular, (2.8) may be rewritten in this case as

$$\frac{1}{2} \int (f(x) - f(y))^2 \, p(x) \, p(y) \, dx dy \leqslant \int \, |\nabla f|^2 \, p(x) \, dx. \tag{2.9}$$

There is a standard argument, which allows one to get on the basis of (2.9) similar Poincaré-type inequalities for perturbed probability measures. Namely, assume that ν has density

$$q(x) = e^{-\frac{1}{2}\,|x|^2 - v(x)},$$

where now the function v is not necessarily convex, but satisfies the condition (2.1) of Theorem 2.1. Then

$$1 \leqslant \int e^{-\frac{1}{2}\,|x|^2 - v^*(x)} \, dx \leqslant e^c,$$

so

$$p(x) = e^{-\frac{1}{2}\,|x|^2 - v^*(x) - c^*}$$

represents a density of some probability measure, say μ, for a suitable constant $c^* \in [0, c]$. It is clear that

$$e^{-c} \, q(x) \leqslant p(x) \, \leqslant e^c \, q(x)$$

for all x, and since p satisfies (2.9), we obtain immediately

$$\frac{e^{-2c}}{2} \int (f(x) - f(y))^2 \, p(x) \, p(y) \, dx dy \leqslant e^c \int \, |\nabla f|^2 \, p(x) \, dx.$$

Equivalently,

$$\mathbf{E}_\nu f^2 - (\mathbf{E}_\nu f)^2 \leqslant e^{3c} \, \mathbf{E}_\nu \, |\nabla f|^2,$$

where the expectations are now taken with respect to ν. This is, however, weaker than the Poincaré-type inequality (2.8).

A similar argument, using linearization of the entropy functional (also leading to a worse behavior of the constant as a function of the parameter c) may be applied to get an analogue of the logarithmic Sobolev inequality (2.7) if we start from the particular case $c = 0$. However, no direct argument seems to properly work if we wish to reach the functional inequality (2.3) on the basis of the same inequality for the class of probability measures, having log-concave densities with respect to γ_n.

Therefore, a different approach is needed to prove Theorems 2.1 and 2.2. We discuss separately the one-dimensional case in Section 3. Then we describe a general localization principle, which is needed to reduce Theorem 2.2 to dimension one, and make final steps of the proof (Section 4). In Section 5, some elementary computations are performed to illustrate the perturbed isoperimetric inequality.

3. One-Dimensional Case

In dimension one, Theorem 2.2 is obtained by combining the Gaussian case in (2.3), when $\mu = \gamma_1$, with the following lemma.

Lemma 3.1. *Let μ be a probability measure on an open interval $\Delta \subset \mathbf{R}$ with density*

$$\frac{d\mu(x)}{dx} = e^{-|x|^2/2 - v(x)},$$

where v is a continuous function on Δ such that

$$v^*(x) \leqslant v(x) \leqslant v^*(x) + c, \quad x \in \Delta, \tag{3.1}$$

for some convex function $v^* : \Delta \to \mathbf{R}$ and a constant $c \geqslant 0$. Then μ represents a contraction of the Gaussian measure on $\mathbf{R}$ with mean zero and variance e^{2c}.

It should be clear that the smallest possible value of c in (3.1) is given by

$$c = \sup_{x \in \Delta} \, [v(x) - v^*(x)], \tag{3.2}$$

where v^* is the convex envelope of v on Δ, i.e.,

$$v^*(x) = \sup\{\ell(x) : \ell \text{ is affine}, \ v \geqslant \ell \text{ on } \Delta\}, \quad x \in \Delta.$$

By the lemma, there is a map $T : \mathbf{R} \to \Delta$ with Lipschitz constant $\|T\|_{\mathrm{Lip}} \leqslant e^c$, which transforms the standard Gaussian measure γ_1 to the measure μ. We will make use of the one-dimensional inequality (2.3) in the case $\mu = \gamma_1$ [**10**]. It gives that, for any locally Lipschitz $u : \mathbf{R} \to [0, 1]$,

$$I\Big(\int u \, d\gamma_1\Big) \leqslant \int \sqrt{I(u)^2 + |u'|^2} \, d\gamma_1.$$

Applying this inequality to the function $u = f(T)$ with $f : \mathbf{R} \to [0, 1]$ locally Lipschitz, and using

$$|u'| \leqslant e^c \, |f'(T)|,$$

where $|u'|$ and $|f'|$ may be understood in the generalized sense according to the definition (2.4), we arrive at

$$I\Big(\int f(T) \, d\gamma_1\Big) \leqslant \int \sqrt{I(f(T))^2 + e^{2c} \, |f'(T)|^2} \, d\gamma_1.$$

Since the distribution of $f(T)$ under γ_1 coincides with the distribution of f under the measure μ, we get

$$I\Big(\int f \, d\mu\Big) \leqslant \int \sqrt{I(f)^2 + e^{2c} \, |f'|^2} \, d\mu. \tag{3.3}$$

This is the desired inequality (2.3) of Theorem 2.2 in dimension one.

Proof of Lemma 3.1. First, let us reformulate the condition (3.1). It is equivalent to the property that, for each point $x_0 \in \Delta$, there is an affine function ℓ such that

(a) $v(x_0) = \ell(x_0)$,

(b) $v(x) \geqslant \ell(x) - c$ for all $x \in \Delta$.

Indeed, assume that, for each point $x_0 \in \Delta$, there is an affine function $\ell = \ell_{x_0}$ with the properties (a)–(b). Then the function

$$v^*(x) = \sup_{x_0 \in \Delta} [\ell_{x_0}(x) - c]$$

is convex and satisfies (3.1).

Conversely, if (3.1) is fulfilled, by the convexity of v^*, for any $x_0 \in \Delta$ there is a tangent affine function ℓ^* to v^* at this point. Put

$$\ell = \ell^* + c^*,$$

where $c^* = v(x_0) - v^*(x_0)$. Then ℓ is the required affine function: $\ell(x_0) = v(x_0)$ and $\ell \leqslant v^* + c \leqslant v + c$ on Δ.

Now, rewrite the density of μ as

$$q(x) = \varphi(x)\, e^{-v(x)}$$

(where the new v differs from the original one by a summand) and denote by

$$F(x) = \mu((-\infty, x]) = \int_{-\infty}^{x} q(y)\, dy$$

the corresponding distribution function. It is strictly increasing and continuous on the supporting interval $\Delta = (a, b)$ of the measure μ. Introduce the converse function $F^{-1} : (0,1) \to \Delta$ and put $q = 0$ outside Δ.

The next argument is similar to the one in [**11**]. The conclusion of Lemma 3.1 may equivalently be stated as the inequality

$$q(F^{-1}(t)) \geqslant e^{-c}\varphi(\Phi^{-1}(t)) = e^{-c} I(t) \quad \text{for all } t \in (0,1).$$

In a different manner, given $x_0 \in \Delta$ and $t \in (0,1)$, if

$$\mu(-\infty, x_0) \geqslant t. \quad \mu(x_0, +\infty) \geqslant 1 - t, \tag{3.4}$$

then

$$q(x_0) \geqslant e^{-c} I(t). \tag{3.5}$$

Thus, fix a point $x_0 \in \mathbf{R}$ and a number $t \in (0,1)$. We will establish (3.5) for the larger class M of all finite positive Borel measures μ on the real line with densities

$$q(x) = \varphi(x) e^{-v(x)},$$

where $v : \mathbf{R} \to (-\infty, +\infty]$ is an arbitrary function such that

$$\Delta = \{x \in \mathbf{R} : v(x) < +\infty\}$$

represents an open interval, where v is continuous and satisfies the conditions (3.1) and (3.4). Note that necessarily $x_0 \in \Delta$.

We want to minimize the quantity $q(x_0)$ as a functional on M, i.e., to find or estimate from below

$$\varkappa = \inf_{\mu \in M} q(x_0).$$

Introduce the subclass $M_0 \subset M$ of all finite positive measures μ on the real line with densities

$$q(x) = \varphi(x)\, e^{-\ell(x)},$$

where ℓ is affine, and also put

$$\varkappa_0 = \inf_{\mu \in M_0} q(x_0).$$

Now, take μ in M with its own q, v, and Δ, and assume that (3.1) is fulfilled on the interval Δ. Hence there is an affine function ℓ with properties (a)–(b). Consider the measure $\overline{\mu}$ on the real line with density

$$\overline{q}(x) = \varphi(x)\, e^{-(\ell(x) - c)}.$$

By property (b),

$$\overline{q}(x) \geqslant q(x) \quad \text{for all } x \in \mathbf{R},$$

and therefore, $\overline{\mu}$ satisfies (3.4), so $\overline{\mu} \in M_0$. By property (a),

$$\overline{q}(x_0) = e^c q(x_0).$$

It follows that

$$\varkappa \geqslant e^{-c} \varkappa_0.$$

Thus, to prove (3.5) and therefore the lemma, it remains to show that $\varkappa_0 \geqslant I(t)$.

For any measure $\mu \in M_0$, its density may be written as

$$q(x) = A\varphi(x) e^{\lambda x} = A e^{\lambda^2/2} \varphi(x - \lambda)$$

with some constants $A > 0$ and $\lambda \in \mathbf{R}$. Since

$$\int_{-\infty}^{x_0} \varphi(x) e^{\lambda x}\, dx = e^{\lambda^2/2} \Phi(x_0 - \lambda),$$

$$\int_{x_0}^{+\infty} \varphi(x) e^{\lambda x}\, dx = e^{\lambda^2/2} (1 - \Phi(x_0 - \lambda)),$$

the condition (3.4) turns into

$$A e^{\lambda^2/2} \Phi(x_0 - \lambda) \geqslant t, \quad A e^{\lambda^2/2} (1 - \Phi(x_0 - \lambda)) \geqslant 1 - t,$$

under which we need to show that

$$q(x_0) = A e^{\lambda^2/2} \varphi(x_0 - \lambda) \geqslant I(t).$$

Equivalently, replacing $B = A e^{\lambda^2/2}$, $y = x_0 - \lambda$, we need to see that

$$B\, \Phi(y) \geqslant t, \quad B\, (1 - \Phi(y)) \geqslant 1 - t \implies B\varphi(y) \geqslant I(t).$$

This is the same as the inequality

$$\max \left\{ t\, \frac{\varphi(y)}{\Phi(y)},\, (1 - t)\, \frac{\varphi(y)}{1 - \Phi(y)} \right\} \geqslant I(t) \quad \text{for all } y \in \mathbf{R}. \tag{3.6}$$

Note that the function $\Phi(y)$ is log-concave, which follows, for example, from the log-concavity of the measure γ_1. Hence

$$(\log \Phi(y))' = \frac{\varphi(y)}{\Phi(y)}$$

is a non-increasing function and attains its minimum in the interval $y \leqslant y_0 = \Phi^{-1}(t)$ at the endpoint y_0. Thus,

$$t\,\frac{\varphi(y)}{\Phi(y)} \geqslant t\,\frac{\varphi(y_0)}{\Phi(y_0)} = I(t), \quad y \leqslant y_0.$$

This proves (3.6) for all $y \leqslant y_0$. Since the function $1 - \Phi(y)$ is log-concave, as well, a similar argument applies to the interval $y \geqslant y_0$. Lemma 3.1 is proved. □

4. Localization. Proof of Theorem 2.2

Reduction of Theorem 2.2 to dimension one uses the localization lemma of Lovász and Simonovits [**16**] (cf. also [**17, 18**] for further developments of the method itself). More precisely, we need a little modified version of the localization lemma, given in [**19**, Corollary 2.4].

Lemma 4.1. *Let R and S be continuous functions on a bounded open convex set Ω in $\mathbf{R}^n$, integrable with respect to the Lebesgue measure and such that*

$$\int R(x)\,dx = 0, \qquad \int S(x)\,dx > 0. \tag{4.1}$$

Then one can find vectors $a, b \in \Omega$ and a log-concave function ψ on $[0,1]$ such that

$$\int_0^1 R(ta + (1-t)b)\,\psi(t)\,dt = 0, \qquad \int_0^1 S(ta + (1-t)b)\,\psi(t)\,dt > 0. \tag{4.2}$$

Moreover, ψ can be chosen to be of the form

$$\psi(t) = \ell(t)^{n-1}$$

for some nonnegative affine function ℓ on $[0,1]$.

In [**19**], this statement is formulated under the assumption that S is lower semi-continuous and bounded. However, the argument leading to the proof shows that we may get rid of the boundedness of S at the expense of the continuity (since the inequalities (2.2) in [**19**, p. 546] remain valid for a smaller open convex subset Ω' of Ω with closure in Ω, and then S will be bounded on Ω').

Now, let μ be an absolutely continuous probability measure on an open convex set $\Omega \subset \mathbf{R}^n$ with density

$$p = e^{-|x|^2/2 - v(x)}, \quad x \in \Omega.$$

Assume that v is continuous on Ω. Given vectors $w, \theta \in \mathbf{R}^n$, $|\theta| = 1$, and a compactly supported log-concave function ψ on $\mathbf{R}$, let us call the one-dimensional density

$$q(s) = \frac{1}{Z}\,p(w + s\theta)\,\psi(s), \quad s \in \mathbf{R}, \tag{4.3}$$

a *generalized conditional distribution* of μ, where Z is a normalizing factor, so that

$$\int q(s)\,ds = 1$$

(and where we may also assume that $w + s\theta \in \Omega$ for all s from the compact support of ψ).

Lemma 4.2. *Assume that every generalized conditional distribution of μ represents a contraction of the Gaussian measure on the real line with mean zero and variance σ^2. Then for any* *locally Lipschitz function f on $\mathbf{R}^n$ with values in* $[0,1]$

$$I(\mathbf{E}f) \leqslant \mathbf{E}\sqrt{I(f)^2 + \sigma^2\,|\nabla f|^2}. \tag{4.4}$$

The expectations are understood with respect to the measure μ.

Proof. First let us note that Ω may be assumed to be bounded. Otherwise, apply the inequality (4.4) to the normalized restrictions μ_n of μ to $\Omega_n = \Omega \cap B(0,n)$ and let $n \to \infty$. Then if (4.4) holds for μ_n, in the limit it will hold for μ, as well. Here, we used the property that every generalized conditional distribution of μ_n represents a generalized conditional distribution of μ.

Thus, assume that Ω is bounded. It is enough to derive (4.4) for the class of continuously differentiable functions $f : \mathbf{R}^n \to [0,1]$ with bounded partial derivatives. Let us formulate (4.4) in a different manner: For any $\alpha \in (0,1)$ and any f

$$\text{if } \mathbf{E}f = \alpha, \text{ then } I(\alpha) \leqslant \mathbf{E}\sqrt{I(f)^2 + \sigma^2\,|\nabla f|^2}.$$

Fix a number $\alpha \in (0,1)$ and (in order to get a contradiction) assume that the above implication does not hold, i.e.,

$$\mathbf{E}f = \alpha, \quad I(\alpha) > \mathbf{E}\sqrt{I(f)^2 + \sigma^2\,|\nabla f|^2}.$$

Equivalently,

$$\mathbf{E}\,(f - \alpha) = 0, \quad \mathbf{E}\left[I(\alpha) - \sqrt{I(f)^2 + \sigma^2\,|\nabla f|^2}\,\right] > 0,$$

so that (4.1) is fulfilled for

$$R(x) = (f(x) - \alpha)\,p(x),$$

$$S(x) = \left[I(\alpha) - \sqrt{I(f(x))^2 + \sigma^2\,|\nabla f(x)|^2}\,\right]p(x).$$

Both functions are continuous on Ω. Hence, by Lemma 4.1, the inequality (4.2) is fulfilled for some vectors $a, b \in \Omega$ and a log-concave function ψ on $[0,1]$.

The case $a = b$ in (4.2) is impossible since it would lead to $R(a) = 0$, $S(a) > 0$, i.e.,

$$f(a) - \alpha = 0, \quad I(\alpha) - \sqrt{I(f(a))^2 + \sigma^2\,|\nabla f(a)|^2} > 0,$$

which is the same as

$$\sqrt{I(\alpha)^2 + \sigma^2\,|\nabla f(a)|^2} < I(\alpha).$$

Thus, necessarily $a \neq b$. Put

$$\theta = \frac{a-b}{|a-b|}, \quad w = b$$

and make the change of the variable $s = |a-b|t$ in order to rewrite (4.2) as

$$\begin{aligned} &\int_0^{|a-b|} R(w + s\theta)\,\widetilde{\psi}(s)\,ds = 0, \\ &\int_0^{|a-b|} S(w + s\theta)\,\widetilde{\psi}(s)\,ds > 0, \end{aligned} \tag{4.5}$$

where

$$\widetilde{\psi}(s) = \psi\Big(\frac{s}{|a-b|}\Big).$$

Note that this function is supported and is log-concave on the interval $\Delta = (0, |a-b|)$. Moreover, in terms of the probability measure ν on Δ with density $q(s)$ defined in (4.3) for the function $\widetilde{\psi}$, (4.5) takes the form

$$\int (f(w + s\theta) - \alpha)\, d\nu(s) = 0, \tag{4.6}$$

$$\int \Big[I(\alpha) - \sqrt{I(f(w + s\theta))^2 + \sigma^2\, |\nabla f(w + s\theta)|^2} \,\Big]\, d\nu(s) > 0. \tag{4.7}$$

Now, the function $g(s) = f(w + s\theta)$ is differentiable on the real line and has derivative

$$g'(s) = \langle \nabla f(w + s\theta), \theta \rangle\,,$$

so

$$|g'(s)| \leqslant |\nabla f(w + s\theta)|.$$

Hence (4.6) and (4.7) imply

$$\int (g - \alpha)\, d\nu = 0,$$

$$\int \Big[I(\alpha) - \sqrt{I(g)^2 + \sigma^2\, |g'|^2} \,\Big]\, d\nu > 0,$$

and all together

$$I\Big(\int g\, d\nu\Big) > \int \sqrt{I(g)^2 + \sigma^2\, |g'|^2}\, d\nu. \tag{4.8}$$

But this contradicts to the assumption that ν represents a contraction of the Gaussian measure on the real line with mean zero and variance $\sigma^2 = e^{2c}$. Indeed, for such measures according to Lemma 3.1 we have the inequality (3.3), i.e., (4.8) with the opposite sign. Lemma 4.2 is proved.

Proof of Theorem 2.2. Let

$$p(x) = e^{-|x|^2/2 - v(x)}, \quad x \in \Omega,$$

be the density of μ. Any generalized conditional distribution of μ has density the form

$$\begin{aligned} q(s) &= \frac{1}{Z}\, p(w + s\theta)\, \psi(s) \\ &= \frac{1}{Z}\, \exp\Big\{ -\frac{s^2}{2} - \langle w, \theta \rangle\, s - \frac{|w|^2}{2} - v(w + s\theta) - V(s) \Big\}, \quad s \in \mathbf{R}, \end{aligned}$$

where $\psi(s) = e^{-V(s)}$ with convex $V : \mathbf{R} \rightarrow (-\infty, +\infty]$. One may restrict this density to the open interval Δ on the real line, where V is finite and $w + s\theta \in \Omega$. Hence

$$q(s) = e^{-\frac{1}{2}\, s^2 - \overline{v}(s)}, \quad \text{where} \quad \overline{v}(s) = v(w + s\theta) + \overline{V}(s), \tag{4.9}$$

with a (finite) convex function $\overline{V}$ on Δ.

Now, by the basic assumption (2.1), for all $s \in \Delta$

$$v^*(w + s\theta) + \overline{V}(s) \,\leqslant\, v(w + s\theta) + \overline{V}(s) \,\leqslant\, v^*(w + s\theta) + \overline{V}(s) + c.$$

It means that the condition (3.1) of Lemma 3.1 is fulfilled for the function $\overline{v}$. Hence, by Lemma 3.1, the measure with density q represents a contraction of the Gaussian measure on the real line with mean zero and variance e^{2c}.

It remains to apply Lemma 4.2 with $\sigma^2 = e^{2c}$.

5. Examples and Remarks

As the above proof shows, Theorem 2.2 may be stated under a slightly weaker assumption on the function v. It might be reasonable to use the following definition.

Definition. Let us say that a function $v : \mathbf{R}^n \to (-\infty, +\infty]$ is *c-quasiconvex* $(c \geqslant 0)$ if $\Omega = \{x : v(x) < +\infty\}$ represents an open convex set in $\mathbf{R}^n$ and for each point $x_0 \in \Omega$ and a line L, passing through x_0, there is an affine function ℓ such that

(a) $v(x_0) = \ell(x_0)$,

(b) $v(x) \geqslant \ell(x) - c$ for all $x \in \Omega \cap L$.

One may also say that v is c-quasiconvex on Ω.

In dimension one, this definition is equivalent to the property (3.1). Hence v is c-quasiconvex on $\Omega \subset \mathbf{R}^n$, if and only if, for any line L such that $\Omega \cap L \neq \varnothing$,

$$(v_L)^* \leqslant v \leqslant (v_L)^* + c \quad \text{on } \Omega \cap L,$$

where $(v_L)^*$ is the convex envelope on L of the restriction $v|_L$ to the interval $\Delta = \Omega \cap L$.

In particular, the optimal value of c, for which v is c-quasiconvex, satisfies

$$c \leqslant \sup_x \, [v(x) - v^*(x)],$$

where v^* is the convex envelope of v on Ω. In dimension one, we have equality, but it is not clear whether it is also true for $n \geqslant 2$.

Anyway, the inequality (2.3) remains valid as long as v is c-quasiconvex on Ω. Moreover, Theorem 2.2 may formally be generalized by comparing μ with non-standard Gaussian measures like in the following.

Theorem 5.1. *Let μ be a probability measure on $\mathbf{R}^n$ with density*

$$\frac{d\mu(x)}{dx} = e^{-|x|^2/2\sigma^2 - v(x)}, \tag{5.1}$$

where $\sigma > 0$ is a parameter and v is a c-quasiconvex function on $\mathbf{R}^n$ $(c \geqslant 0)$. Then for any locally Lipschitz function f on $\mathbf{R}^n$ with values in $[0,1]$

$$I\Big(\int f \, d\mu\Big) \leqslant \int \sqrt{I(f)^2 + C^2 |\nabla f|^2} \, d\mu, \quad C = \sigma e^c. \tag{5.2}$$

Let us mention a few examples, where one can easily compute the optimal value of c on the basis of the one-dimensional formula (3.2).

1. Assume that a continuous function v is convex on the half-axis $(-\infty, x_1]$ and is convex on the half-axis $[x_2, +\infty)$ for some $x_1 < x_2$. Let ℓ be the affine function whose graph passes through the points $(x_1, v(x_1))$, $(x_2, v(x_2))$. Assume that the graph of v lies above the graph of ℓ on the interval $[x_1, x_2]$. Then v is c-quasiconvex, where

$$c = \max_{x_1 \leqslant x \leqslant x_2} [v(x) - \ell(x)].$$

2. If, in addition, the function v is even on the real line and concave on the interval $[-x_2, x_2]$, then $c = v(0) - v(x_2)$.

3. For example, for the polynomial

$$v(x) = Ax^4 + Bx^2$$

with $A > 0$ we have $c = 0$ if $B \geqslant 0$. In the other case,

$$v''(x) = 12Ax^2 + 2B = 0 \Leftrightarrow x_{1,2} = \pm\sqrt{\frac{-B}{6A}},$$

so $c = \dfrac{5B^2}{36A}$.

4. More generally, consider an arbitrary polynomial of degree 4,

$$v(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E \quad \text{with } A > 0.$$

The affine part $Dx + E$ has no influence on c. To reduce computations to the previous example, note that after a proper shifting we have

$$v\left(x - \frac{B}{4A}\right) = Ax^4 + \left(C - \frac{3B^2}{8A}\right)x^2 + \text{ affine part.}$$

Hence

$$c = 0 \quad \text{if } C \geqslant \frac{3B^2}{8A},$$

in which case v is convex, and

$$c = \frac{5\,(C - \frac{3B^2}{8A})^2}{36A} \quad \text{if} \quad C \leqslant \frac{3B^2}{8A}.$$

5. Given parameters $p > 2$ and $\sigma > 0$, consider the function

$$v(x) = \frac{1}{p}\,|x|^p - \frac{1}{2\sigma^2}\,|x|^2.$$

It is easy to see in this case,

$$c = \frac{p-2}{2p}\,\sigma^{-\frac{2p}{p-2}}. \tag{5.3}$$

Formula (5.3) continues to hold for a similar function v on $\mathbf{R}^n$ (for the Euclidean norm $|\cdot|$). It may be used to derive an inequality of the form

$$I\left(\int f\,d\mu_p\right) \leqslant \int \sqrt{I(f)^2 + C_p^2\,|\nabla f|^2}\,d\mu_p \tag{5.4}$$

for the spherically invariant probability measure μ_p on $\mathbf{R}^n$ with density

$$\frac{d\mu_p(x)}{dx} = \frac{1}{Z}\,e^{-\frac{1}{p}\,|x|^p}, \quad x \in \mathbf{R}^n,$$

where Z is a normalizing constant (which depends both on p and n). Put $\sigma = 1$ and represent the density of μ_p in the form (5.1), in which case $c = \dfrac{p-2}{2p}$, according to (5.3). In view of (5.2), we then have the following assertion.

Corollary 5.2. *For any locally Lipschitz function* $f : \mathbf{R}^n \to [0,1]$ *the inequality* (5.4) *holds for the measure* μ_p, $p \geqslant 2$, *with constant*

$$C_p^2 = \exp\{\frac{p-2}{p}\}.$$

Note that $C_2 = 1$, which corresponds to the Gaussian case $\mu_2 = \gamma_n$, and $C_p \leqslant e$ for all $p \geqslant 2$. An inequality similar to (5.4) may also be obtained by noting that μ_p represents a transform of γ_n under a map having a finite Lipschitz constant. This route involves some routine computations or estimation of the Lipschitz constant, while the approach based on the perturbations (Theorem 2.2) seems to be much simpler.

In the case $1 \leqslant p < 2$, which was studied by many authors, the inequality (5.4) is no longer true for the Gaussian isoperimetric function I (cf., for example, [**20, 21, 22, 23**], where the isoperimetric problem and related functional inequalities were considered for the measures μ_p on the real line and their products μ_p^n on $\mathbf{R}^n$).

Note also that, in some examples, Theorem 5.1 is more suitable in comparison with Theorem 2.2 due to the flexible parameter $\sigma > 0$ (which can be used for the optimization of the constant C in (5.2)).

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